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On Characteristic Poset and Stanley Decomposition

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Abstract

Let $J \subset I$ be two monomial ideals such that I/J is Cohen Macaulay. By associating a finite posets $P^g_{I/J}$ to I/J, we show that if I/J is a Stanley ideal then $\widetilde{I/J}$ is also a Stanley ideal, where $\widetilde{I/J}$ is the polarization of I/J. We also give relations between sdepth and fdepth of I/J and $\widetilde{I/J}$.

Introduction

Let K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables. Let $J \subset I \subset S$ be two monomial ideals such that I/J is a $\mathbb{Z}^n$-graded S-module. Let $u \in I/J$ be a homogeneous monomial and $Z \subseteq \{x_1, \ldots, x_n\}$. We denote by $uK[Z]$ the K-subspace of I/J generated by all elements uv where v is a monomial in $K[Z]$. If $uK[Z]$ is a free $K[Z]$-module then the $\mathbb{Z}^n$-graded K-subspace $uK[Z] \subset I/J$ is called a *Stanley space of dimension* $|Z|$.

A *Stanley decomposition* of I/J is a presentation of the $\mathbb{Z}^n$-graded K-vector space I/J as a finite direct sum of Stanley spaces $\mathcal{D} : I/J = \bigoplus_{i=1}^m u_i K[Z_i]$. The *Stanley depth* of I/J is defined to be

$$\operatorname{sdepth}(I/J) = \max\{\operatorname{sdepth} \mathcal{D} \colon \mathcal{D} \text{ is a Stanley decomposition of } I/J\}.$$

Stanley [10] conjectured that there always exists a Stanley decomposition such that $\operatorname{sdepth}(I/J) \geq \operatorname{depth}(I/J)$. If the conjecture holds for some ideal I, then

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we call I a Stanley ideal. The conjecture is widely discussed in recent years for example [2], [4], [6], [8], [9].

Let u be a monomial in S. Then

$$\tilde{u} = \prod_{i=1}^{n} \prod_{j=1}^{a_i} x_{ij} \in T$$

is called the *polarization* of u, where $T = K[x_{11}, \ldots, x_{1a_1}, \ldots, x_{n1}, \ldots, x_{na_n}]$. Let I be a monomial ideal in S with monomial generators $(u_1, \ldots, u_r)$. Then the ideal generated by $(\tilde{u}_1, \ldots, \tilde{u}_r)$ is called the *polarization* of I and is denoted by $\tilde{I}$. For more details, see [3]. As a main result of this paper we show that if I/J is a CM Stanley ideal, then $\widetilde{I/J}$ is also a CM Stanley ideal by using a more appropriate approach than [1]. We use the idea of *characteristic poset* from [6]. A partial order on $\mathbb{N}^n$ is given by $(a(1), \ldots, a(n)) \leq (b(1), \ldots, b(n))$ if $a(i) \leq b(i)$ for all i. Let $g \in \mathbb{N}^n$ be an integer vector with the property that $a \leq g$ for all $a \in \mathbb{Z}^n$ with $x^a \in I/J$. Here x^a denote the monomial $x_1^{a(1)} \ldots x_n^{a(n)}$ where $a = (a(1), \ldots, a(n)) \in \mathbb{N}^n$. The *characteristic poset* (see [6]) $P_{I/J}^g$ of I/J with respect to g is the subposet of $\mathbb{N}^n$ given by

$$P_{I/J}^g = \{a \in \mathbb{Z}^n : \ x^a \in I \setminus J, \quad a \leq g\}.$$

Each Stanley decomposition of I/J gives a partition of $P_{I/J}^g$ and vice versa. We call a partition of $P_{I/J}^g$, a *nice* partition if its corresponding Stanley decomposition satisfies Stanley's conjecture. In Proposition 1.2, we give a necessary and sufficient condition for a partition to be nice. In Theorem 1.5, we show that if $P_{I/J}$ has a nice partition, then $P_{\widetilde{I/J}}$ also has a nice partition.

In [6], the concept of fdepth is introduced which is a natural lower bound for sdepth and depth. It is defined as

$$\operatorname{fdepth} M = \max\{\operatorname{fdepth} \mathcal{F} : \ \mathcal{F} \text{ is a prime filtration of } M\}.$$

In Corollary 1.6, we show that sdepth (fdepth) of $\widetilde{I/J}$ can be computed by computing sdepth (fdepth) of I/J.

1 Posets and their Partitions

The natural partial order $\leq$ on $\mathbb{N}^n$ is defined as follows: $a \leq b$, with $a = (a(1), \ldots, a(n))$ and $b = (b(1), \ldots, b(n))$ if and only if $a(i) \leq b(i)$ all for $i = 1, \ldots, n$. The meet $a \wedge b$ and join $a \vee b$ with respect to $\leq$ are $(\min\{a(1), b(1)\}, \ldots, \min\{a(n), b(n)\})$ and $(\max\{a(1), b(1)\}, \ldots, \max\{a(n), b(n)\})$,

respectively.
With this natural partial order $\mathbb{N}^n$ is a distributive lattice.

Let K be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables. For any $c = (c(1), \ldots, c(n)) \in \mathbb{N}^n$ we denote by x^c the monomial $x_1^{c(1)} \cdots x_n^{c(n)}$. Let I and J be two monomial ideals in S such that $J \subset I$. Let $I = (x^{a_1}, \ldots, x^{a_r})$ and $J = (x^{b_1}, \ldots, x^{b_s})$ for some $a_i, b_i \in \mathbb{N}^n$ for all i. We associate a poset to I/J in the following way: we choose $g \in \mathbb{N}^n$ such that $a_i \leq g$ and $b_j \leq g$ for all i and j. Let $P^g_{I/J}$ be the set of all $c \in \mathbb{N}^n$ with $c \leq g$ and such that $a_i \leq c$ for some i and $c \ngeq b_j$ for all j. The set $P^g_{I/J}$ is a finite subposet in $\mathbb{N}^n$ and called (see [6]) the *characteristic poset* of I/J with respect to g. A natural choice for g is the join of all the a_i and b_j. For this choice of g, the poset $P^g_{I/J}$ has the least number of elements.

Given any poset P and $a, b \in P$, an *interval* $[a, b]$ is defined as $[a, b] = \{c \in P\colon\ a \leq c \leq b\}$. Suppose P is a finite poset. A *partition* of P is a disjoint union

$$\mathcal{P}\colon\ P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals of P.

In order to describe the Stanley decomposition of I/J coming from a partition of $P^g_{I/J}$ we adopt the following notation from [6]: for each $b \in P^g_{I/J}$, Z_b is the set $\{x_j\colon\ b(j) = g(j)\}$. The function ρ is introduced as

$$\rho\colon\ P^g_{I/J} \to \mathbb{Z}_{\geq 0}, \quad c \mapsto \rho(c),$$

where $\rho(c) = |\{j\colon\ c(j) = g(j)\}| (= |Z_c|)$. Now we quote the following theorem from [6].

Theorem 1.1. *Let* $\mathcal{P}\colon P^g_{I/J} = \bigcup_{i=1}^{r} [c_i, d_i]$ *be a partition of* $P^g_{I/J}$. *Then*

$$\mathcal{D}(\mathcal{P})\colon I/J = \bigoplus_{i=1}^{r} \Big(\bigoplus_{c} x^c K[Z_{d_i}]\Big) \tag{1}$$

is a Stanley decomposition of I/J*, where the inner direct sum is taken over all* $c \in [c_i, d_i]$ *for which* $c(j) = c_i(j)$ *for all* j *with* $x_j \in Z_{d_i}$. *Moreover,* $\operatorname{sdepth} \mathcal{D}(\mathcal{P}) = \min\{\rho(d_i)\colon\ i = 1, \ldots, r\}$.

It is also shown in [6, Theorem 2.4] that $\operatorname{sdepth} I/J$ can be computed as the maximum of the numbers $\operatorname{sdepth} \mathcal{D}(\mathcal{P})$, where $\mathcal{P}$ runs over the (finitely many) partitions of $P^g_{I/J}$. From these results we conclude that Stanley's conjecture holds for I/J if and only if there exists a partition $\mathcal{P} : P^g_{I/J} = \bigcup\limits_{i=1}^{t} [a_i, b_i]$ of the poset P such that

$$|\rho(b_i)| \geq \operatorname{depth}(I/J) \quad \text{for all} \quad i. \tag{2}$$

Any partition of a poset satisfying condition (1.2) will be called *nice*.

To this end, we give some definitions associated to a poset. Let P be a finite poset. An element $m \in P$ is called a *maximal* element if there is no $a \in P$ with $a > m$. We denote by $\mathcal{M}(P)$ the set of maximal elements of P. An element $a \in P$ is called a *facet* of P if for all $m \in \mathcal{M}(P)$ with $a \leq m$ one has $\rho(a) = \rho(m)$. The set of all facets of P will be denoted by $\mathcal{F}(P)$.

A chain $\mathfrak{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of $\mathbb{Z}^n$-graded submodules of M is called a *prime filtration* of M if $M_i/M_{i-1} \cong (S/P_i)(-a_i)$ where $a_i \in \mathbb{Z}^n$ and where each P_i is a monomial prime ideal. The set of prime ideals $\{P_1, \ldots, P_m\}$ is called the *support* of $\mathfrak{F}$ and denoted by $\operatorname{supp}\mathfrak{F}$.

The next proposition gives a necessary and sufficient condition for a partition $\mathcal{P}$ of $P^g_{I/J}$ to be nice.

Proposition 1.2. *Let $J \subset I$ be two monomial ideals of S such that I/J is Cohen-Macaulay. Let $P^g_{I/J}$ be the poset associated to I/J and $\mathcal{P} : P^g_{I/J} = \bigcup_{i=1}^t [a_i, b_i]$ be a partition of $P^g_{I/J}$. Then the following conditions are equivalent.*

(a) $\mathcal{P}$ is nice.

(b) $\{b_1, \ldots, b_t\} \subseteq \mathcal{F}(P^g_{I/J})$

(c) $\mathcal{M}(P^g_{I/J}) \subseteq \{b_1, \ldots, b_t\} \subseteq \mathcal{F}(P^g_{I/J})$

Proof. (a) $\Rightarrow$ (b): Since I/J is Cohen-Macaulay therefore $|\rho(b)| \leq \operatorname{depth}(I/J)$ for all faces of $P^g_{I/J}$, and $|\rho(b)| = \operatorname{depth}(I/J)$ if and only if b is a facet. Thus $\mathcal{P}$ is nice only if $\{b_1, \ldots, b_t\} \subseteq \mathcal{F}(P^g_{I/J})$.

(b) $\Rightarrow$ (c): Let $m \in \mathcal{M}(P^g_{I/J})$ and since $\mathcal{P}$ is partition of $P^g_{I/J}$, so $m \in [a_i, b_i]$ for some i. Since $m \leq b_i$ and m is maximal, it follows that $m = b_i$. Thus $\mathcal{M}(P^g_{I/J}) \subseteq \{b_1, \ldots, b_t\}$.

(c) $\Rightarrow$ (a): Let $\mathfrak{F}$ be a prime filtration of I/J. Then $\{\dim S/P : P \in \operatorname{Ass}(I/J))\} = \{\rho(b_i) : b_i \in \mathcal{F}(P^g_{I/J})\}$. Therefore,

$$\begin{aligned}
\min\{\rho(b_i)\colon b_i \in \mathcal{F}(P^g_{I/J})\} &= \min\{\rho(m_j)\colon m_j \in \mathcal{M}(P^g_{I/J})\} \\
&= \min\{\dim(S/P_j)\colon P_j \in \operatorname{Ass}(I/J)\} \\
&\geq \operatorname{depth}(I/J).
\end{aligned}$$

The first equation follows from the definition of the facets, while the last inequality is a basic fact of commutative algebra. Therefore our given partition is nice. $\square$

Remark 1.3. In the above Proposition if $\mathcal{P}$ is nice then we can refine it in such a way that for the refinement

$$\mathcal{P}' : P^g_{I/J} = \bigcup_{i=1}^{t'} [a'_i, b'_i]$$

we have $\{b'_1, \dots, b'_{t'}\} = \mathcal{F}(P^g_{I/J})$. To prove this fact, let $\hat{b}$ be a facet such that $\hat{b} \neq b'_i$, for all i. So there exist some interval $[a_i, b_i]$ with $\hat{b} \in [a_i, b_i]$. Since $\hat{b}$ is a facet, we obtain $\rho(\hat{b}) = \rho(b_i)$, and there exist j such that $\hat{b}(j) < b_i(j)$. We set $a'_i(j) = \hat{b}(j) + 1$ and $a'_i(k) = a_i(k)$ for all k different from j. The interval $[a_i, b_i]$ in $\mathcal{P}$ can be replaced by two disjoint intervals, namely $[a_i, \hat{b}]$ and $[a'_i, b_i]$ and obtain the desired conclusion.

Let u be a monomial in S. Then $\tilde{u} = \prod_{i=1}^{n} \prod_{j=1}^{a_i} x_{ij} \in T$ is called the *polarization* of u, where $T = K[x_{11}, \dots, x_{1a_1}, \dots, x_{n1}, \dots, x_{na_n}]$. Let $I \subset S$ be a monomial ideal such that $I = (u_1, \dots, u_r)$. Then the ideal generated by $(\tilde{u}_1, \dots, \tilde{u}_r)$ is called a *polarization* of I and is denoted by $\tilde{I}$. We may assume that for each $i \in [n]$ there exists j such that x_i divides u_j. Let $u_j = x_1^{a_{j1}} \cdots x_n^{a_{jn}}$ for $j = 1, \dots, s$ and set $r_i = \max a_{ji} : \ j = 1, \dots, s$ for $i = 1, \dots, n$. Moreover we set $r = \sum_{i=1}^{n} r_i$. Then $\tilde{I}$ is a squarefree monomial ideal in the polynomial ring T in r variables. It is known that I is Cohen-Macaulay if and only if $\tilde{I}$ is Cohen-Macaulay.

We denote the posets associated with I/J and $\widetilde{I/J}$ by $P^g_{I/J}$ and $P^{g'}_{\widetilde{I/J}}$, and the set of facets of $P^g_{I/J}$ and $P^{g'}_{\widetilde{I/J}}$ by $\mathcal{F}(P^g_{I/J})$ and $\mathcal{F}(P^{g'}_{\widetilde{I/J}})$. Note that $\mathcal{F}(P^g_{I/J})$ is a subset of the set

$$B = \{b \in \mathbb{N}^n : b(i) < r_i \text{ if } b(i) \neq g(i)\}.$$

In order to formulate the main theorem of this paper, we introduce the following notion. We define the map $\varphi : B \to N^r$ as follows:

$$b \mapsto b'(ij) = \begin{cases} 0, & \text{if } b(i) < g(i) \text{ and } j = b(i) + 1, \\ g'(ij), & \text{otherwise.} \end{cases}$$

The components of the vectors b' are indexed by pairs of numbers ij and for each $i = 1, \dots, n$ the second index j runs in the range $j = 1, \dots, r_i$.

The set of facets of a poset can be viewed as the set of facets of a multicomplex Γ in the following way: if a is a facet of Γ with $a(i) = \infty$, then we set $a(i) = g(i)$. For details about multicomplex, see [5]. From the result of Soleyman Jahan [7, Proposition 3.8], we see that the restriction of the map φ

to $\mathcal{F}(P^g_{I/J})$ is a bijection between the set of facets of $P^g_{I/J}$ and the set of facets of $P^{g'}_{\widetilde{I/J}}$.

Consider the set $A = \{a \in \mathbb{N}^n : a(i) \leq r_i\}$ and the map $\psi : A \to \{0,1\}^r$ defined as

$$\psi(a)(ij) = \begin{cases} 0, & \text{if } j > a(i), \\ 1, & \text{otherwise.} \end{cases}$$

The map ψ is injective, because if for any $a \neq a'$ there exists some i such that $a(i) \neq a'(i)$, say $a(i) < a'(i)$. Then by definition of ψ, $a(ij) = 0$ for $j = a(i)+1$ and $a'(ij) = 1$ for $j = a(i) + 1$.

Let $I = [a, b] \subset \mathbb{N}^n$ be an interval such that $a = (a(1), a(2), \dots, a(n))$ and $b = (b(1), b(2), \dots, b(n))$. An *i-subinterval* of I is defined as $\{c \in \mathbb{N} : a(i) \leq c \leq b(i)\}$ and is denoted it by $I(i) = [a(i), b(i)]$. Next we quote the following Lemma from [1].

Lemma 1.4. *Let I_1, I_2 be two intervals of a poset P such that $I_1 = [a, b]$ and $I_2 = [c, d]$. Suppose $I_1 \cap I_2 = \emptyset$, then $I_1(i) \cap I_2(i) = \emptyset$, for some $i \in \{1, \dots, n\}$*

Now we state the following theorem.

Theorem 1.5. *Let $I \subset J \subset S$ be two monomial ideals of S such that I/J is Cohen-Macaulay and $\widetilde{I/J}$ be the polarization of I/J. Also let $P^g_{I/J}$ and $P^{g'}_{\widetilde{I/J}}$ be the characteristic posets associated to I/J and $\widetilde{I/J}$, respectively. If $P^g_{I/J}$ has a nice partition then $P^{g'}_{\widetilde{I/J}}$ has also a nice partition.*

Proof. Let $P^g_{I/J}$ has a nice partition say $\mathcal{P}'$. As described in Remark 1.3 we can refine the partition $\mathcal{P}'$ to a refined partition say $\mathcal{P} := \bigcup_{i=1}^t [c_i, d_i]$ such that $\{d_1, \dots, d_t\} = \mathcal{F}(P)$. We will show that $\hat{\mathcal{P}} := \bigcup_{i=1}^t [\hat{c}_i, \hat{d}_i]$ is a nice partition of $P^{g'}_{\widetilde{I/J}}$, where $\psi(c_i) = \hat{c}_i$ and $\varphi(d_i) = \hat{d}_i$ for all $i = 1, \dots, t$.

First, we show that $[\hat{c}_i, \hat{d}_i] \cap [\hat{c}_j, \hat{d}_j] = \emptyset$, for all $i \neq j$. Suppose that there exist a face $c \in [\hat{c}_i, \hat{d}_i] \cap [\hat{c}_j, \hat{d}_j] \neq \emptyset$ for some $i \neq j$. Injectivity of ψ gives $\hat{c}_i \neq \hat{c}_j$. Since $[c_i, d_i] \cap [c_j, d_j] = \emptyset$, we apply Lemma 1.4 and obtain $[c_i(l), d_i(l)] \cap [c_j(l), d_j(l)] = \emptyset$ for some $l \in \{1, \dots, n\}$. It shows that at least one of $d_i(l), d_j(l)$ is not equal to $g(l)$ say $d_i(l) \neq g(l)$. Then by Theorem 1.1 $b_i(l) = a_i(l)$. If $d_j(l) = g(l)$ and $c_i(l) > c_j(l)$ then $[c_i(l), d_i(l)] \subset [c_j(l), d_j(l)]$ which is not possible so we may assume that $c_i(l) < c_j(l)$. On the other hand if $d_j(l) \neq g(l)$, then we may change i by j. Thus we can assume let $c_i(l) = d_i(l) = k-1$ and $c_j(l) = m > k-1$. Then by definition of $P^g_{I/J}$ and φ we have $\hat{c}_i(lk) = 0 = \hat{d}_i(lk)$ and $\hat{c}_j(ll) = 1$ for $l \leq m$. Thus $\hat{c}_j(lk) = 1$. It follows that $c(lk) = 0$. On the other hand, since $c \geq \hat{c}_j$, we get $c(lk) \geq \hat{c}_j(lk) = 1$ and we obtain a contradiction to our assumption that $[\hat{c}_i, \hat{d}_i] \cap [\hat{c}_j, \hat{d}_j] \neq \emptyset$.

Now for the second part of the proof, we will use the Hilbert series. We have $H(S/I) = \sum_{i=1}^{t} s^{|c_i|}/(1-s)^{\rho(d_i)}$. The definition of ψ implies that $\rho(c_i) = \rho(\hat{c}_i)$ for all $i = \{1, \ldots, t\}$. We know that the depth of I/J increases by 1 for each polarization step. Also, we observe from the definition of φ that for each polarization step, $\rho(d_i)$ increases by 1. Therefore, after p polarization steps $\rho(\hat{d}_i) = \rho(d_i) + p$ and

$$H(\bigcup_{i=1}^{t}[\hat{c}_i, \hat{d}_i]) = \sum_{i=1}^{t} \frac{s^{\rho(c_i)}}{(1-s)^{\rho(d_i)+p}} = \frac{1}{(1-s)^p} H(S/I)$$

is the Hilbert series of $H(\widetilde{I/J})$. Hence $\widetilde{I/J} = \bigcup_{i=1}^{t}[\hat{c}_i, \hat{d}_i]$. Note that $\bigcup_{i=1}^{t}[\hat{c}_i, \hat{d}_i]$ is a nice partition because $\rho(\hat{d}_i) = \rho(d_i) + p \geq \operatorname{depth}_S(S/I) + p = \operatorname{depth}_T(\widetilde{I/J})$, for all i. □

The converse of Theorem 1.5 is still open. We recall the definition of fdepth from [6]. Let $\mathfrak{F}$ be a prime filtration of I/J. Furthermore, $\operatorname{fdepth} \mathfrak{F} = \min\{\dim S/P\colon\ P \in \operatorname{supp} \mathfrak{F}\}$ and

$$\operatorname{fdepth} M = \max\{\operatorname{fdepth} \mathfrak{F}\colon\ \mathfrak{F} \text{ is a prime filtration of } M\}.$$

It is not obvious how to compute the fdepth of a module, but it is very easy to see that $\operatorname{fdepth} M \leq \operatorname{depth} M, \operatorname{sdepth} M$.

Corollary 1.6. *With same notation as above, we have*

(a) $\operatorname{sdepth}(\widetilde{I/J}) = \operatorname{sdepth}(I/J) + r - n$,

(b) $\operatorname{fdepth}(\widetilde{I/J}) \geq \operatorname{fdepth}(I/J)$.

Where r and n are the number of variables of T and S, respectively.

Proof. Let $c \in \mathcal{F}(P^g_{I/J})$ and $\varphi(c) \in \mathcal{F}(P^{g'}_{\widetilde{I/J}})$. Since sdepth(fdepth) of a Stanley decomposition of I/J is minimum of the numbers $\rho(c)$, the assertion follows by observing that $\rho(\varphi(c)) = \rho(c) + r - n$. □

References

[1] S. Ahmad, *Stanley decompositions and polarization*, Czechoslovak Mathematical Journal, vol. 61, no. 2 (2011), pp. 483-493.

[2] I. Anwar and D. Popescu, *Stanley conjecture in small embedding dimension*, J. Alg. **318**(2007), 1027-1031. Zbl 1132.13009

[3] J. Herzog, T. Hibi, *Monomial Ideals*, Springer, 2011.

[4] J. Herzog, A. Soleyman Jahan, S. Yassemi, *Stanley decompositions and partitionable simplicial complexes*, J. Algebraic Combinatorics, 27 (2008), 113-125. Zbl 1131.13020

[5] J. Herzog, D. Popescu, *Finite filtrations of modules and shellable multicomplexes*, Manuscripta Math. **121**, (2006), 385-410. Zbl 1107.13017

[6] J. Herzog, M. Vladoiu and X. Zheng, *How to compute the Stanley depth of a monomial ideal*, Journal of Algebra, 322 (2009), 3151-3169. Zbl pre05658663

[7] A. Solyman Jahan, *Prime filtrations of monomial ideals and polarizations*, J. Algebra 312(2007), 1011-1032.

[8] S. Nasir, *Stanley decomposition and localization*, Bull. Math. Soc. Sc. Math. Roumanie **51**(99), no.2(2008), 151-158.

[9] D. Popescu, *Stanley depth of Multigraded modules*, J. Algebra 321(2009), 2782-2797. Zbl 1179.13016

[10] R. P. Stanley, *Linear Diophantine equations and local cohomology*, Invent. Math.**68**, (1982), 175-193. Zbl 0516.10009

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