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Known results and open problems in hypercomplex convexity

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Abstract

A subject, which is treated in this review, combines in one bundle some questions of convex, hypercomplex analysis, probability theory and geometry.

In the following, we shall start with the known definition of the convexity, but further we shall generalize this definition on more broad classes of sets in Euclidean spaces.

All terms, which are used in this article without determination can be found in [1].

Definition 1. A set E in a real Euclidean space $\mathbb{R}^n$ is called *convex*, if for every pair of points a and b from E the closed interval $[a, b]$ is subset of E.

It is easy make sure that this definition is equivalent to the following.

Definition 1a. A set E in a real Euclidean space $\mathbb{R}^n$ is called *convex*, if an intersection of E with any real line l is connected.

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The previous definitions reflect the internal characteristics of a set. The following definition is external. It, according to the geometric form of the Hahn-Banach theorem, satisfies the convex domains and compacts.

Definition 2. A set $E \subset \mathbb{R}^n$ is called *generalized convex*, if for every point $x \in \mathbb{R}^n \backslash E$ there exists a hyperplane L such that $x \in L \subset \mathbb{R}^n \backslash E$.

Example 1. Let $E = \{(x, y) \in \mathbb{R}^2 : (x < 0) \vee (0, 0)\}$ be an open half-plane with one point on boundary.

This set satisfies the first definition, but does not satisfy the second. Every line through the point $a = (0, y)$, where $y \neq 0$, intersects the set E.

Example 2. Let $E = \{(x, y) \in \mathbb{R}^2 : (1 < |x| < 4) \vee (1 < |y| < 4)\}$ be the set of four open squares.

This set satisfies the second definition, but does not satisfy the first. As it is remarked in [1] each component of a set determined by definition 2 will be a convex set and union of arbitrary two components of such set will also satisfy definition 2. The last remark remains true for union of two components only. From the following example is seen that union of arbitrary three components of example 2 already does not satisfy the definition 2.

Example 3. Let we consider set E_1 equal to the set E of the example 2 without the square from forth quadrants. Each line through arbitrary point of the triangle

$$T = \{(x, y) \in \mathbb{R}^2 : (x < -1) \wedge (y < -1) \wedge (x - y < 3)\}$$

intersects the set E_1.

Generalizing this construction, the approach is a consideration a family of the linear submanifolds of Euclidean space, which crosses the given set. In the real case this question is well studied [2]. The case of complex, hypercomplex and more general space, as it is noted in [2], has not got the sufficient development yet. The geometric approach to generalized convexity in complex spaces is described in monograph [1]. In hypercomplex case known results are more less.

In the hypercomplex case following two classes will be the natural generalization of the class of convex sets.

We consider $\mathbb{H}^n$ – n-dimensional hypercomplex Euclidean space, $z = (z_1, z_2, \ldots, z_n)$ – a point in $\mathbb{H}^n$.

Definition 3. A set $E \subset \mathbb{H}^n$ is called *linearly convex*, if for every point $z \in \mathbb{H}^n \backslash E$ there exists a left hyperplane l such that $z \in l \subset \mathbb{H}^n \backslash E$.

Example 4. All convex and compact domains are linearly convex.

Example 5. Cartesian product $E = E_1 \times E_2 \times \ldots \times E_n$ of arbitrary sets $E_i \subset \mathbb{H}$ is linearly convex set, in particular, $T = S^3 \times S^3 \times \ldots \times S^3$, where S^3 is three-dimensional sphere, is linearly convex.

Hence obviously that set, satisfying Definition 3, it is enough complicated and can powerfully differ from convex set.

The following definition draws near us to class of convex sets.

Definition 4. A set $E \subset \mathbb{H}^n$ is called $\mathbb{H}$-*convex*, if for every hypercomplex line γ intersection $\gamma \cap E$ is acyclic.

The next result installs one side relationship between two classes of generalized convexity [3].

Theorem 1. *All $\mathbb{H}$-convex compacts are linearly convex.*

First of all, the concept of linear convexity in complex space $\mathbb{C}^2$ was introduced in 1935 in the paper of Behnke H. and Peschl E. [4] and was used widely by A.Martineau and L.Aizenberg from the sixties of the last century [5, 6].

Linearly convex sets are very useful in complex analysis and in the issues of the integral geometry and tomography. On the base of these sets in complex analysis there is built linearly convex complex analysis, similar to real convex analysis. More results of linearly convex analysis can be viewed in the next monographs [1, 7-9].

In spite of abundance of results, a big amount of the unsolved problems have remained, concerning topological characteristics of these sets, a part from which it is possible to find in [1, 8, 10]. It seems very interesting for the author simply formulated the following open problem of the sphere.

Problem 1. (Sphere problem). Is there a linearly convex compact in complex Euclidean space $\mathbb{C}^2$, for which all cohomology groups coincide with the corresponding cohomology groups of the two-dimensional sphere S^2?

Some of the problems of this theme are connected with the known Ulam problem from the Scottish book [11].

Ulam problem. Let M^n be n-dimensional manifold and every section of M^n by hyperplanes L be homeomorphic to $(n-1)$-dimensional sphere S^{n-1}. Is it true that M^n is n-dimensional sphere?

In the real case this problem is solved by A. Kosiński in 1962 [12]. The repetition of this result was obtained by L. Montejano in 1990 [13]. In complex case similar result was obtained by Yu. Zelinskii in 1993 [7].

Other problems of this group are to find: analogically to Ulam problem an estimation of the properties of a set if we know the properties of its intersections with the families of some sets:

1) with the planes of a fixed dimension:

i) in the real case (G. Auman, A. Kosinski, E. Shchepin [14,12,15]);

ii) in the complex case (Yu. Zelinskii [7]);

2) with a set of vertices of an arbitrary rectangle (A.Besicowitch, L.Danzer, T.Zamfirescu, M.Tkachuk [16-19]).

The last problem is known in literature as Mizel problem.

Mizel problem (Characterization of a circle). Let C be a convex Jordan curve on a plane $\mathbb{R}^2$ with the following property:

For every rectangle *abcd*, if any three vertices are on C, then the fourth vertex will be also on C. Is it true that C is a circle?

This problem was solved by Besicovitch and Danzer independently [16, 17].

In 2006, my PhD-student M. Tkachuk [19] obtained the most general result in this area for arbitrary compact $K \subset \mathbb{R}^2$, where $\mathbb{R}^2 \setminus K$ is not connected.

The row of similar opened problems in the plane and in n-dimensional case appear in connection with the Mizel's problem.

In 2012, Yu. Zelinskii, M. Tkachuk, B. Klishchuk obtained positive answer on one problem of Zamfirescu related to the Misel's problem [18, 20].

Theorem 2. *The convex curve of constant width satisfying the infinitesimal rectangular condition is a circle.*

The following results of the author, his colleagues and students describe some properties of generalized convex sets with additional assumptions.

Theorem 3 [3]. *Any section of bounded linearly convex domain with smooth boundary in H^n by hypercomplex line is simpleconnected and have connected boundary for $n > 1$.*

Theorem 4 [1,7]. *Let $K \subset \mathbb{H}^n$ be linearly convex compact, then the cohomology groups $H^i(K) = 0$ for all $i > 3n$.*

We remark that for compact T from example 2 $H^{3^n}(T) \neq 0$.

Theorem 5 [20]. *A compact, which is a non degenerated Cartesian product in $\mathbb{H}^n$, is $\mathbb{H}$-convex if and only if it is convex.*

We remark that a Cartesian product $A \times B$ is non degenerated if nor A nor B are point.

Definition 5. A domain D in a hypercomplex Euclidean space $\mathbb{H}^n$ is called *locally linearly convex*, if for every point z of the boundary D there exists a left hyperplane l and neighbourhood $U(z)$ then $z \in l$ and $D \cap l \cap U(z) = \varnothing$.

Let $\rho(z)$ be a function of n hypercomplex variable and we consider a domain $\Omega = \{z : \rho(z) < 0\}$ in the space $\mathbb{H}^n$ with the smooth boundary $\partial\Omega = \{z : \rho(z) < 0\}$ of the class C^2.

The next result describes local properties of generalized convex domains, with the smooth boundary.

Theorem 6 [23]. *For a domain Ω be locally linearly convex it is necessary that for every point from the boundary next inequalities holds*

$$\sum_{i,j=1}^{n} \sum_{k,l=0}^{3} \frac{\partial^2 \rho(w)}{\partial z_i^k \partial z_j^l} s_j^l s_i^k \geq 0,$$

$$\sum_{i,j=1}^{n} \sum_{k,l=0}^{3} s_i^k \, \frac{\partial^2 \rho(w)}{\partial z_i^k \partial z_j^l} s_j^l \geq 0,$$

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} s_j^l s_i^k \ \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} \geq 0,$$

for every tangent to boundary at point w non zero vectors $s = (s_1, s_2, \ldots, s_n)$ and sufficient that one of the following strong inequalities holds

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} s_j^l s_i^k > 0,$$

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} s_i^k \ \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} s_j^l > 0,$$

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} s_j^l s_i^k \ \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} > 0$$

for the same vectors s.

Now we shall give the row of the opened problems of this theme.

Problem 1a. (Sphere problem in hypercomplex space). Is there a linearly convex compact in $\mathbb{H}^2$, for which all cohomology groups coincide with the corresponding cohomology group of some sphere S^m, where $4 \leq m \leq 6$?

Problem 2. Let $K \subset \mathbb{H}^n$ be a linearly convex compact for which cohomology group $H^j(K) \neq 0,\ j \geq 4$. Is this true that there exists i, $1 \leq i < j$, that $H^i(K) \neq 0$?

Problem 3. Is this true or not that every bounded locally linearly convex domain in $\mathbb{H}^n$ with smooth boundary, $n > 1$ is $\mathbb{H}$-convex and homeomorphicaly equivalent to $4n$-dimensional ball in Euclidean space $\mathbb{R}^{4n}$.

Problem 4. What are characteristics of cohomology groups for hypercomplex holomorphe domains? Is this true or not that cohomology groups $H^i(D) = 0$ for all $i > 3n$? (Compare with the theorem 4).

Problem 5. The questions to approximations of linearly convex sets by sets of the same class, but additionally with smooth or nearly smooth boundary.

Problem 6. The description of strong linearly convex compacts by their extreme and foreseeable points of the boundary.

Problem 7. (**The shadow problem**). What is minimal number of pair wise disjoint balls with the centre on sphere S^{n-1} it is enough that any straight line, getting through the centre of the sphere, crossed at least one of those balls?

Problem 8. A characterization of curves and surfaces by some properties of their intersection with algebraic curves and surface of the fixed order.

Problem 9. The description hyperspace of linearly convex sets and thick subset in him. Here are desired effects, look like studies L. Bazylevych [25], but main difficulty of the complex case, unlikely to real analysis, is in nonlinear structure of such hyperspaces.

With other opened problems, connected with Mizel's problem and generalized convexity, is possible look in [1, 20, 24].

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