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A duality-type method for the obstacle problem

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Abstract

Based on a duality property, we solve the obstacle problem on Sobolev spaces of higher order. We have considered a new type of approximate problem and with the help of the duality we reduce it to a quadratic optimization problem, which can be solved much easier.

1 Introduction

The study of the obstacle problem has a long history. One of the first authors who treated the obstacle problem is G. Fichera, [8, 9]. Later on, J. Frehse, [10], proved that the second derivatives of the solution are bounded by using the minimum principle of superharmonic functions. In the work of D.G. Schaeffer, [18], 1975, the obstacle problem is solved using the Nash-Moser implicit function theorem.

We also quote the monographs of D. Kinderlehrer and G. Stampacchia, [14], R. Glowinski, [12] and V. Barbu, T. Precupanu, [4], devoted or including consistent investigations on unilateral problems.

The high interest in the obstacle problem is due to its multiple applications such as the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control, and financial mathematics (C. Baiocchi.[3], G. Duvaut, J.-L. Lions [7] and P. Wilmott, S. Howison, J. Dewynne, [19]).

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Some recent articles in this subject are R. Griesse, K. Kunisch, [11], C. M. Murea, D. Tiba [16], M. Burger, N. Matevosyan, M.T Wolfram, [6].

The obstacle problem is often referred to in papers that develop new algorithms as an important example for testing them. For instance, L. Badea, [2], one- and two-level domain decomposition methods are tested on a two obstacles problem, in higher order Sobolev spaces. And speaking of algorithms, many authors are still developing special tools to solve the obstacle problem (F. A. Pérez, J. M. Cascón, L. Ferragut, [17]).

In this paper, we are discussing the obstacle problem defined on the Sobolev space $W_0^{1,p}(\Omega)$, for $p > dim\ \Omega$. The idea we use is that of solving the problem with the help of an approximate problem of its dual, which we state in Section 2 and that seems to be new. We apply the Fenchel's duality theorem, in Section 3 and we analyze the dual problem that appears from the application of the duality theorem. Using another duality argument, K. Ito, K. Kunisch, [13], introduced the primal-dual active set strategies and they proved that the method based on this strategy is equivalent to a semi-smooth Newton method. In this paper we show that the solution of the approximate dual problem is a linear combination of Dirac distributions. Finally, we are able to treat the approximate obstacle problem by simply solving a quadratic minimization problem and applying a formula which transfers the result back to the primal approximate problem. In Section 4 we apply the algorithm to the one dimensional obstacle problem. The results of this paper have been announced in the note [15] (without proofs).

2 The problem and its approximation

We consider that $\Omega \subset \mathbb{R}^d$ is a bounded open set with the strong local Lipschitz property. We study the obstacle problem

$$\min_{y \in W_0^{1,p}(\Omega)_+} \left\{ \frac{1}{2} \|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \right\} \tag{1}$$

where $f \in L^1(\Omega)$, $p > d = dim\ \Omega$, and $W_0^{1,p}(\Omega)_+ = \{y \in W_0^{1,p}(\Omega) : y \geq 0 \text{ in } \Omega\}$.

By the Sobolev theorem, we have $W^{1,p}(\Omega) \to C(\overline{\Omega})$ and it makes sense to consider the following approximate problem

$$\min \left\{ \frac{1}{2} \|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \ \ : \ y \in W_0^{1,p}(\Omega); y(x_i) \geq 0, i = 1, 2, \ldots, k \right\} \tag{2}$$

where $\{x_i\}_{i \in \mathbb{N}} \subseteq \Omega$ is a dense set in Ω. For each $k \in \mathbb{N}$, we denote

$$C_k = \{y \in W_0^{1,p}(\Omega) : y(x_i) \geq 0, i = 1, 2, \ldots, k\}$$

the closed convex cone.

We have

Proposition 1. *The following assertions are true*

(i) *Problem* (1) *has a unique solution* $\bar{y} \in W_0^{1,p}(\Omega)_+$.

(ii) *Problem* (2) *has a unique solution* $\bar{y}_k \in C_k$.

This is an easy consequence of the compact imbedding $W^{1,p}(\Omega) \to L^\infty(\Omega)$, which follows from the Rellich-Kondrachov Theorem (R. Adams [1], Theorem 6.2, Part II, page 144).

Moreover, we can prove the following approximation result

Theorem 1. *The sequence* $\{\bar{y}_k\}_k$ *of the solutions of problems* (2), *for* $k \in \mathbb{N}$, *is a strongly convergent sequence in* $W^{1,p}(\Omega)$ *to the unique solution* $\bar{y}$ *of the problem* (1).

Proof. Let $\{\bar{y}_k\}_{k\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ be the sequence of the solutions of the problems (2). Consider $y \in W_0^{1,p}(\Omega)_+$ arbitrary. Then $y \in C_k$, for every $k \in \mathbb{N}$. It follows that

$$\frac{1}{2}\|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \geq \frac{1}{2}\|\bar{y}_k\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega f\bar{y}_k, \quad \forall y \in W_0^{1,p}(\Omega)_+, \forall k \in \mathbb{N}. \quad (3)$$

Then

$$\inf_{y \in W_0^{1,p}(\Omega)_+} \left\{\frac{1}{2}\|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy\right\} \geq \frac{1}{2}\|\bar{y}_k\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega f\bar{y}_k, \quad \forall k \in \mathbb{N}.$$

Knowing that $\inf(P) = M < +\infty$, with $M > 0$ constant, it yields

$$\begin{aligned} M &\geq \frac{1}{2}\|y_k\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy_k \\ &\geq \frac{1}{2}\|\bar{y}_k\|^2_{W_0^{1,p}(\Omega)} - c\|f\|_{L^1(\Omega)}\|\bar{y}_k\|_{W_0^{1,p}(\Omega)}. \end{aligned}$$

So the sequence $\left\{\|\bar{y}_k\|_{W_0^{1,p}(\Omega)}\right\}_k$ is bounded. Thus the sequence $\{\bar{y}_k\}_k \in W_0^{1,p}(\Omega)$ is weakly convergent to an element $\hat{y} \in W_0^{1,p}(\Omega)$, on a subsequence.

Since $\bar{y}_k(x_i) \geq 0$ and $\bar{y}_k \to \hat{y}$ uniformly on $\overline{\Omega}$, then for every $x \in \overline{\Omega}$ we have $\bar{y}_k(x) \to \hat{y}(x)$. Then $\hat{y}(x_i) \geq 0$, $\forall i \in \mathbb{N}$. Considering that the set $\{x_i : i \in \mathbb{N}\}$ is assumed to be dense in $\overline{\Omega}$, it results that $\hat{y} \in W_0^{1,p}(\Omega)_+$. In conclusion, $\hat{y}$ is admissible for (1).

On the other hand, since $\bar{y} \in W_0^{1,p}(\Omega)_+$, we can write (3) for $\bar{y}$, which means that

$$\frac{1}{2}\|\bar{y}\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega f\bar{y} \geq \frac{1}{2}\|\bar{y}_k\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega f y_k. \tag{4}$$

Passing to the limit, and considering the weak inferior semicontinuity of the norm, we obtain

$$\frac{1}{2}\|\bar{y}\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega f\bar{y} \geq \frac{1}{2}\|\hat{y}\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega f\hat{y}.$$

But, since problem (1) has a unique solution, it follows that $\bar{y} = \hat{y}$. So, we have proved that $\bar{y}_k \to \bar{y}$ weakly in $W_0^{1,p}(\Omega)$.

For the strong convergence, we use (4) and get that

$$\frac{1}{2}\|\bar{y}\|^2_{W_0^{1,p}(\Omega)} \geq \limsup_{k\to\infty} \frac{1}{2}\|y_k\|^2_{W_0^{1,p}(\Omega)}. \tag{5}$$

By the weak convergence already proven we get

$$\frac{1}{2}\|\bar{y}\|^2_{W_0^{1,p}(\Omega)} \leq \liminf_{k\to\infty} \frac{1}{2}\|y_k\|^2_{W_0^{1,p}(\Omega)}. \tag{6}$$

Then, it follows, from (5) and (6) that $\bar{y}_k \to \bar{y}$ strongly in $W_0^{1,p}(\Omega)$, using Proposition 3.32, page 78, H. Brezis, [5]. The convergence is valid without taking subsequences since the limit is unique. □

3 The dual problem

In this section we shall use the dual of the problem (2) to solve problem (1).

We apply Fenchel's duality Theorem to obtain the dual problems associated to problems (1) and (2). For this purpose we consider the functional

$$F(y) = \frac{1}{2}\|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy, \quad y \in W_0^{1,p}(\Omega). \tag{7}$$

Let q be the exponent conjugate of p. Using the definition of the convex conjugate and the fact that the duality mapping $J : W_0^{1.p}(\Omega) \to W^{-1,q}(\Omega)$ is a single-valued and bijective operator, we get that the convex conjugate of F is

$$F^*(y^*) = \frac{1}{2}\|f + y^*\|^2_{W^{-1,q}(\Omega)}$$

The argument is similar with the one used for $p = 2$ in V. Barbu, Th. Precupanu, [4], page 192.

Considering now the functional $g = -I_{W_0^{1,p}(\Omega)_+}$ and using the concave conjugate definition we get that

$$g^{\bullet}(y^*) = \begin{cases} 0, & y^* \in (W_0^{1,p}(\Omega)_+)^* \\ -\infty, & y^* \notin (W_0^{1,p}(\Omega)_+)^* \end{cases}$$

with $(W_0^{1,p}(\Omega)_+)^* = \{y^* \in W^{-1,q}(\Omega) : (y, y^*) \geq 0, \forall y \in W_0^{1,p}(\Omega)_+\} = W^{-1,q}(\Omega)_+$.

Since F and $-g$ are convex and proper functionals on $W^{1,p}(\Omega)$, the domain of g is $D(g) = W_0^{1,p}(\Omega)_+$, and F is continuous everywhere on $W_0^{1,p}(\Omega)_+$ we are able to apply Fenchel duality Theorem (V. Barbu, Th. Precupanu, [4], pp 189) and obtain

$$\min_{y \in W_0^{1,p}(\Omega)_+} \left\{ \frac{1}{2} \|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \right\} =$$
$$\max \left\{ -\frac{1}{2} \|f + y^*\|^2_{W^{-1,q}(\Omega)} : y^* \in W^{-1,q}(\Omega)_+ \right\}.$$

The dual problem associated to problem (1) is

$$\max \left\{ -\frac{1}{2} \|f + y^*\|^2_{W^{-1,q}(\Omega)} : y^* \in W^{-1,q}(\Omega)_+ \right\}.$$

For the approximate problem (2) we only need the concave conjugated of $g_k = -I_{C_k}$ due to the fact that we minimize the same functional F over another cone. Thus, the concave conjugate is

$$g_k^{\bullet}(y^*) = \inf \{(y, y^*) - g_k(y) : y \in C_k\} = \begin{cases} 0, & y^* \in C_k^* \\ -\infty, & y^* \notin C_k^* \end{cases}$$

where $C_k^* = \{y^* \in W^{-1,q}(\Omega) : (y^*, y) \geq 0, \forall y \in C_k\}$.

Lemma 1. *The polar cone of C_k is*

$$C_k^* = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{x_i} : \alpha_i \geq 0 \right\}$$

where δ_{x_i} are the Dirac distributions concentrated in $x_i \in \Omega$, i.e. $\delta_{x_i}(y) = y(x_i)$, $y \in W_0^{1,p}(\Omega)$.

Proof. We consider

$$\hat{C}_k = \left\{ u = \sum_{i=1}^{k} \alpha_i \delta_{x_i} : \alpha_i \geq 0 \right\}$$

First, the Dirac distributions δ_{x_i} are linear and continuous functionals due to the fact that $W_0^{1,p}(\Omega) \subset C(\overline{\Omega})$. This yields that $\hat{C}_k \subset W^{-1,q}(\Omega)$.

The aim is to compute the polar of the cone $\hat{C}_k$.

By definition of the polar cone, we have

$$\hat{C}_k^* = \left\{ y \in W_0^{1,p}(\Omega) : (y,u) \geq 0, \forall u \in \hat{C}_k \right\}.$$

Since

$$(y,u) = (y, \sum_{i=1}^{k} \alpha_i \delta_{x_i}) = \sum_{i=1}^{k} \alpha_i (y, \delta_{x_i}) = \sum_{i=1}^{k} \alpha_i y(x_i)$$

and $\alpha_i \geq 0, \forall i = \overline{1,k}$ we obtain the equivalence

$$(y,u) \geq 0 \quad \Leftrightarrow \quad y(x_i) \geq 0, \quad \forall i = \overline{1,k}.$$

Then

$$\hat{C}_k^* = \left\{ y \in W_0^{1,p}(\Omega) : y(x_i) \geq 0, \forall i = \overline{1,k} \right\} = C_k$$

This means that $(\bar{C}_k^*)^* = C_k^*$.

By the Theorem of bipolars (V. Barbu, Th. Precupanu, [4], pp 88), we have

$$\hat{C}_k^{**} = \overline{conv(\hat{C}_k \cup \{0\})}. \tag{8}$$

Since $0 \in \hat{C}_k$ and the cone $\hat{C}_k$ is convex, it only remains to be proven that $\hat{C}_k$ is a closed cone.

Consider $u \in \overline{\hat{C}_k}$. Then we can find a sequence $(u_n)_n \in \hat{C}_k$ convergent to u in $W^{-1,q}(\Omega)$. Since $u_n \in \hat{C}_k$, we get

$$u_n = \sum_{i=1}^{k} \alpha_i^n \delta_{x_i} \to u \ in \ W^{-1,q}(\Omega).$$

Let $S(x_i, r) \subset \Omega$ be such that $x_j \notin S(x_i, r)$, for $i \neq j$. For every $i \in \{1, 2, \ldots, k\}$, let $\rho_i \in \mathcal{D}(S(x_i, r)) \subset \mathcal{D}(\Omega)$ such that $\rho_i(x_i) = 1$. Then

$$\left(\sum_{i=1}^{k} \alpha_i^n \delta_{x_i}, \rho_j \right) \to (u, \rho_j), \quad \forall j = \overline{1,k}.$$

We obtain

$$\alpha_j^n \to (u, \rho_j), \quad \forall j = \overline{1,k}.$$

In the end, we denote $\alpha_j = (u, \rho_j), \quad \forall j = \overline{1,k}$.

Thus, from the above arguments, we conclude that

$$u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{i=1}^{k} \alpha_i^n \delta_{x_i} = \sum_{i=1}^{k} \left(\lim_{n\to\infty} \alpha_i^n\right) \delta_{x_i} = \sum_{i=1}^{k} \alpha_i \delta_{x_i}$$

This implies that $u \in \hat{C}_k$. Thus the cone $\hat{C}_k$ is a closed one.

It yields that relation (8) can be rewritten as

$$\hat{C}_k^{**} = \hat{C}_k$$

This shows that $\hat{C}_k = C_k^*$ as claimed. □

Since the domain of g_k is $D(g_k) = C_k$ and the functional F is still continuous on the closed convex cone C_k, the hypothesis of Fenchel duality Theorem are satisfied again. This implies that

$$\begin{aligned} &\min\left\{\frac{1}{2}\|y\|^2_{W_0^{1,p}(\Omega)} - \int_\Omega fy \ \ : \ y \in C_k\right\} = \\ &\max\left\{-\frac{1}{2}\|y^* + f\|^2_{W^{-1,q}(\Omega)} : y^* \in C_k^*\right\} \end{aligned} \tag{9}$$

So we obtain the dual approximate problem associated to problem (2)

$$\max\left\{-\frac{1}{2}\|y^* + f\|^2_{W^{-1,q}(\Omega)} : y^* \in C_k^*\right\}. \tag{10}$$

Theorem 2. *Let $\bar{y}_k$ be the solution of the approximate problem* (2) *and $\bar{y}_k^*$ the solution of the dual approximate problem* (10)*. Then the two solutions are related by the formula*

$$\bar{y}_k = J^{-1}(\bar{y}_k^* + f) \tag{11}$$

where J is the duality mapping $J : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$. Moreover, $(\bar{y}_k^*, \bar{y}_k) = 0$.

Proof. Applying Theorem 2.4 (V. Barbu, Th. Precupanu, [4], pp 188) we get the following system of equations

$$\bar{y}_k^* \in \partial F(\bar{y}_k), \tag{12}$$

$$-\bar{y}_k^* \in \partial I_{C_k}(\bar{y}_k) \tag{13}$$

where the functional F is the functional defined as in (7).

From (12), by using the definition of the subdifferential of a convex function, we obtain $\bar{y}_k^* + f \in J(\bar{y}_k)$. Since this mapping is single-valued and bijective, we get that $\bar{y}_k = J^{-1}(\bar{y}_k^* + f)$.

From (13), using again the definition of the subdifferential, we get

$$I_{C_k}(\bar{y}_k) - I_{C_k}(z) \leq (-\bar{y}_k^*, \bar{y}_k - z), \quad \forall z \in C_k$$

Choosing $z = \frac{1}{2}\bar{y}_k$, it follows that

$$I_{C_k}(\bar{y}_k) \leq -(\bar{y}_k^*, \bar{y}_k)$$

Then, for $z = 2\bar{y}_k \in C_k$, we get the opposite inequality

$$I_{C_k}(\bar{y}_k) \geq -(\bar{y}_k^*, \bar{y}_k)$$

But, since $\bar{y}_k \in C_k$, we can conclude that

$$(y_k^*, y_k) = 0$$

$\square$

Remark 1. *Since $\bar{y}_k^* \in C_k^*$, by Lemma 1, we know*

$$\bar{y}_k^* = \sum_{i=1}^{k} \alpha_i^* \delta_{x_i}$$

where $\alpha_i^* \geq 0$ for all $i = 1, 2, \ldots, k$. then

$$(\bar{y}_k^*, \bar{y}_k) = (\sum_{i=1}^{k} \alpha_i^* \delta_{x_i}, \bar{y}_k) = \sum_{i=1}^{k} \alpha_i^* (\delta_{x_i}, \bar{y}_k) = \sum_{i=1}^{k} \alpha_i^* \bar{y}_k(x_i)$$

Thus,

$$\sum_{i=1}^{k} \alpha_i^* \bar{y}_k(x_i) = 0$$

Again $\bar{y}_k \in C_k$, and this means that $\bar{y}_k(x_i) \geq 0$ for all $i = 1, 2, \ldots, k$. It follows that

$$\alpha_i^* \bar{y}_k(x_i) = 0, \quad \forall i = 1, 2, \ldots, k$$

Then, in conclusion, the Lagrange multipliers α_i^* are zero if $\bar{y}_k(x_i) > 0$ and they can be positive only when the constraint is active, i.e. $\bar{y}_k(x_i) = 0$.

4 Numerical applications

In this section we apply the above theoretical results to solve the obstacle problem (1) in dimension one.

We consider $\Omega = (-1, 1)$ and $p = 2$. The statement of the obstacle problem is

$$\min_{y \in H_0^1(\Omega)_+} \left\{ \frac{1}{2} |y|^2_{H_0^1(\Omega)} - \int_\Omega fy \right\}$$

Using Theorem 1, we write the approximate problem

$$\min_{y \in C_k} \left\{ \frac{1}{2} |y|^2_{H_0^1(\Omega)} - \int_\Omega fy \right\} \tag{14}$$

where $C_k = \{y \in H_0^1(\Omega) : y(x_i) \geq 0, \forall i = 1, 2, \ldots, k\}$. The set $\{x_i : i \in \mathbb{N}\}$ is, as above, dense in Ω.

From the equality (9), we can write the dual approximate problem

$$\min \left\{ \frac{1}{2} |y^* + f|^2_{H^{-1}(\Omega)} : y^* \in C_k^* \right\} \tag{15}$$

where $C_k^* = \{y^* \in H^{-1}(\Omega) : y = \sum_{i=1}^k \alpha_i \delta_{x_i}, \alpha_i \geq 0, \forall i = 1, 2, \ldots, k\}$.

The duality mapping, in this case, is $J : H_0^1(\Omega) \to H^{-1}(\Omega)$ and is define as $J(y) = -y''$. It is a linear bounded operator.

Let $\bar{y}_k$ the solution of problem (14) and $\bar{y}_k^*$ the solution of problem (15). Then, by Theorem 2, we get

$$\bar{y}_k = J^{-1}(\bar{y}_k^* + f).$$

Using the form of an element in C_k^* and the fact that J is linear, we can rewrite the above formula as

$$\bar{y}_k = \sum_{i=1}^k \alpha_i J^{-1}(\delta_{x_i}) + J^{-1}(f).$$

To compute $J^{-1}(\delta_{x_i})$ we consider the following Cauchy problem

$$\begin{cases} d_i' = -H_i + a, & \text{pe } (-1, 1) \\ d_i(-1) = 0 \end{cases} \tag{16}$$

where H_i is the Heaviside function concentrated in x_i, i.e.

$$H_i(x) = \begin{cases} 0, & x < x_i \\ 1, & x > x_i \end{cases}$$

The real constant a is computed such as $d_i(1) = 0$.

Then we get that

$$J^{-1}(\delta_{x_i}) = d_i = \begin{cases} \frac{1}{2}(1-x_i)(x+1), & x \le x_i \\ \frac{1}{2}(1-x_i)(x+1) - (x - x_i), & x > x_i \end{cases}$$

From problem (16) we obtain that $-d_i'' = H_i' = \delta_{x_i}$.

To compute $J^{-1}(f)$ we need to solve the problem

$$\begin{cases} -y_f'' = f, \quad \text{pe } (-1,1) \\ y_f(-1) = y_f(1) = 0 \end{cases}$$

Using the equality

$$|\bar{y}_k|^2_{H_0^1(-1,1)} = |\bar{y}_k^* + f|^2_{H^{-1}(-1,1)} = \Big| \sum_{i=1}^k \alpha_i J^{-1}(\delta_{x_i}) + J^{-1}(f) \Big|^2_{H^{-1}(-1,1)}$$

that the problem (15) can be rewritten as

$$\min \left\{ \frac{1}{2} \Big| \sum_{i=1}^k \alpha_i d_i + y_f \Big|^2_{H_0^1(-1,1)} : \alpha_i \ge 0 \right\}. \tag{17}$$

Define the functional $G : \mathbb{R}^k \to \mathbb{R}$,

$$G(\alpha) = \Big| \sum_{i=1}^k \alpha_i d_i + y_f \Big|^2_{H^{-1}(-1,1)}.$$

Computing the norm we end up with

$$G(\alpha) = \sum_{i,j=1}^k \alpha_i \alpha_j \int_{-1}^1 d_i' d_j' dx + 2 \sum_{i=1}^k \alpha_i \int_{-1}^1 d_i' y_f' dx + \int_{-1}^1 (y_f')^2$$

Let us denote

$$\begin{aligned} a_{ij} &= \int_{-1}^1 d_i' d_j' dx \quad \forall i,j = 1,2,\ldots,k \\ b_i &= \int_{-1}^1 d_i' y_f' dx \quad \forall i = 1,2,\ldots,k \qquad c = \int_{-1}^1 (y_f')^2. \end{aligned}$$

Now considering $A = [a_{ij}]$ and $b = [b_i]$, we can write

$$G(\alpha) = \alpha^T A \alpha + 2 b^T \alpha + c$$

It follows that solving problem (17) is in fact equivalent to solving the following quadratic problem

$$\min_{\alpha\in\mathbb{R}^k_+}\left\{\frac{1}{2}\alpha^T A\alpha + b^T\alpha\right\} \tag{18}$$

All that remains to do is to compute the elements of the matrix A and those of the vector b. This is easily done and we obtain that

$$a_{ij} = \begin{cases} \frac{1}{2}(1+x_i)(1-x_j), & j>i \\ \frac{1}{2}(1+x_j)(1-x_i), & j\le i \end{cases}$$

and

$$b_i = y_f(x_i), \quad \forall i = 1,2,\dots,k$$

Thus, the problem of finding the optimal coefficients α_i^* of the solution $\bar{y}_k^*$ can be done by simply applying the Matlab function `quadprog`.

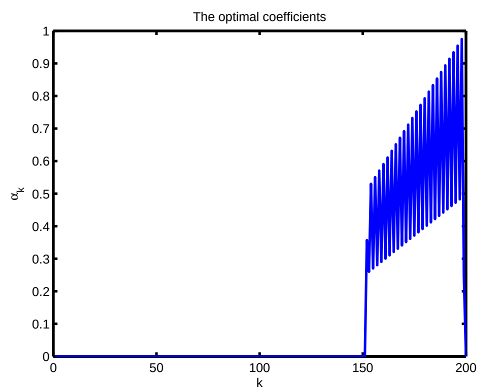

Figure 1: The coefficients $\{\alpha_i^*\}_{i=1}^k$.

Then, using the results of the Theorem 2, we find the solution $\bar{y}_k$ of the approximate problem (14).

Let us consider $k = 200$ and $f = -300x$. We solve (18) and we get the α_i^* coefficients represented in Figure 1

We now compute the approximate problem solution, which is represented in Figure 2.

The same way, we can solve the problem on the interval $\Omega = (0,1)$. We compute again the functions

$$d_i(x) = \begin{cases} (1-x_i)x, & x<x_i \\ x_i(1-x), & x\ge x_i \end{cases}$$

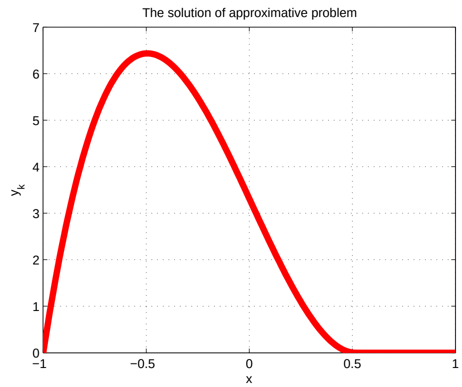

Figure 2: The approximate problem solution

and then the elements of A and b are, in this case,

$$a_{ij} = \begin{cases} x_i(1-x_j), & j > i \\ x_j(1-x_i), & i \geq j \end{cases} \qquad i,j = 1,2,\ldots,k$$

and $b_i = y_f(x_i), \quad i = 1,2,\ldots,k.$

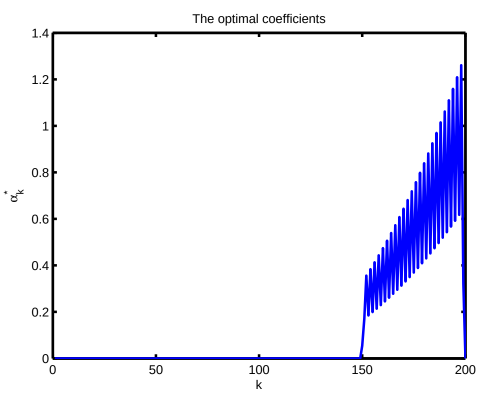

Figure 3: The coefficients $\{\alpha_i^*\}_{i=1}^k$.

Taking again $k = 200$ and considering the function $f(x) = -300x^3 + 100x$, we can compute the optimal coefficients solving problem (18). They are represented in Figure 3.

Consequently, we compute the solution of the approximate problem in the same manner as above and we end up with the solution represented in Figure 4.

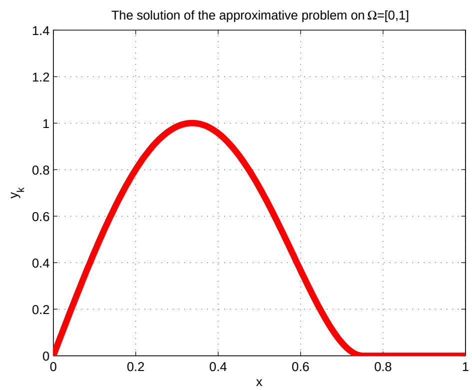

Figure 4: The approximate problem solution

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