



# A Classifier for Unimodular Isolated Complete Intersection Space Curve Singularities

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## Abstract

C.T.C. Wall classified the unimodular complete intersection singularities. He indicated in the list only the  $\mu$ -constant strata and not the complete classification in each case. In this article we give a complete list of space curve unimodular singularities and also the description of a classifier. Instead of computing the normal forms, the singularity is identified by certain invariants.

## 1 Introduction

Marc Giusti gave the complete list of simple isolated complete intersection singularities which are not hypersurfaces (cf. [GM83]). An implementation in SINGULAR for the classification of simple isolated complete intersection singularities over the complex numbers is given by Gerhard Pfister and Deeba Afzal in `classifyci.lib` as a SINGULAR library (cf. [ADPG1], [ADPG2]). Wall achieved the classification of contact unimodular singularities which are not hypersurfaces (cf. [Wal83]).

We report about a classifier for unimodular isolated complete intersection curve singularities in the computer algebra system SINGULAR (cf. [DGPS13],[GP07]). A basis for a classifier is a complete list of these singularities together with a list of invariants characterizing them. Since Wall gave only representatives of the  $\mu$ -constant strata in his classification (cf. [Wal83]), we complete his list by computing the versal  $\mu$ -constant deformation of the

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singularities. The new list obtained in this way contains all unimodular complete intersection curve singularities. In section 2 we characterize the 2-jet of the unimodular complete intersection singularities by using primary decomposition and Hilbert polynomials. In section 3 we give the complete list of unimodular complete intersection space curve singularities by fixing the 2-jet of the singularities and develop algorithms for each case. In section 4 we present examples.

Let us recall the basic definitions.

Let  $\mathbb{C}[[x]] = \mathbb{C}[[x_1, \dots, x_n]]$  be the local ring of formal power series and  $\langle x \rangle = \langle x_1, \dots, x_n \rangle$  its maximal ideal.

**Definition 1.1.**  $f = \langle f_1, f_2, \dots, f_p \rangle$ , is called *complete intersection* if  $\dim \mathbb{C}[[x]]/\langle f_1, \dots, f_i \rangle = n - i, \forall i = 1, \dots, p$ .

*Hypersurfaces* are special cases of complete intersections for  $p = 1$ .

**Definition 1.2.** Let  $f = \langle f_1, \dots, f_p \rangle \subseteq \mathbb{C}[[x]]$  be a complete intersection.  $f = \langle f_1, \dots, f_p \rangle$  has an *isolated singularity* at 0, if

1.  $\langle f_1, \dots, f_p, M_1, \dots, M_k \rangle \subseteq \langle x \rangle$ ,  $M_1, \dots, M_k$  the  $p \times p$ -minors of  $(\frac{\partial f_i}{\partial x_j})$ .
2.  $\langle x \rangle^c \subseteq \langle f_1, \dots, f_p, M_1, \dots, M_k \rangle$  for some  $c > 0$ .

The *Milnor number*  $\mu(f)$  is defined as follows

$$\mu(f) = \sum_{i=1}^p (-1)^{p-i} \dim_{\mathbb{C}} \mathbb{C}[[x]]/C_i$$

with  $C_i = \langle f_1, f_2, \dots, f_{i-1}, \frac{\partial(f_1, \dots, f_i)}{\partial x_{j_1 \dots j_i}}, 1 \leq j_1, \dots, j_i \leq n \rangle$  (cf. [GM75]).

The *Tjurina number* of  $f$  is defined to be

$$\dim_{\mathbb{C}} \mathbb{C}[[x]]^p / f \mathbb{C}[[x]]^p + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathbb{C}[[x]].$$

Let  $I_{n,p}$  be the set of all isolated complete intersection singularities. Then  $G_c = \text{Aut}(\mathbb{C}[[x]]) \times GL_p(\mathbb{C}[[x]])$  acts on  $I_{n,p}$  as follows:

let  $(\phi, \psi) \in G_c$  and  $f = (f_1, \dots, f_p) \in I_{n,p}$  then  $(\phi, \psi)(f) = \psi^{-1} \circ f \circ \phi$ .

**Definition 1.3.** Let  $f$  and  $g \in I_{n,p}$  are called *contact equivalent*, if there exists  $(\phi, \psi) \in G_c$ , such that  $f = (\phi, \psi)(g)$ .

$I_{n,p} \subseteq \mathbb{C}[[x]]^p$  carries a canonical topology. It is the topology such that the maps

$$\mathbb{C}[[x]]^p \rightarrow (\mathbb{C}[[x]]/\langle x \rangle^c)^p$$

are continuous  $\forall c$ . Here we consider the classical topology of the affine space  $(\mathbb{C}[[x]]/\langle x \rangle^c)^p$ .

**Definition 1.4.** An element  $f \in I_{n,p}$  is called *simple singularity*, if there exists a neighborhood of  $f$  in  $I_{n,p}$  containing only finitely many orbits of  $G$ . In other words the modality of the singularity is zero.

**Definition 1.5.**  $f$  is called *unimodular singularity* if there exists a neighborhood of  $f$  in  $I_{n,p}$  containing only one-dimensional families of orbits of  $G_c$ . In other words the modality of the singularity is 1.

**Definition 1.6.**  $f \in I_{n,p}$  defines a *curve* if  $\mathbb{C}[[x]]/f$  is of dimension 1.

If  $f$  defines an irreducible curve, i.e  $f \subseteq \mathbb{C}[[x]]$  is prime. Then the normalization of  $\mathbb{C}[[x]]/f$  is  $\mathbb{C}[[t]]$  and we have parametrization

$$\mathbb{C}[[x]]/f \cong K[[x_1(t), x_2(t), \dots, x_n(t)]] \subseteq \mathbb{C}[[t]]$$

**Definition 1.7.**

$$\Gamma_f = \{ord(f) \mid f \in \mathbb{C}[[x_1(t), x_2(t), \dots, x_n(t)]]\}$$

is the *semi group* of the curve.

**Definition 1.8.**  $f = \langle f_1, \dots, f_p \rangle \in I_{n,p}$  then  $F = \langle F_1, \dots, F_p \rangle$ ,  $F_i \in \mathbb{C}[[x, t]]$  where  $t = \{t_1, \dots, t_n\}$  is a *deformation of  $f$*  if

$$\mathbb{C}[[x]]/\langle f \rangle \cong \mathbb{C}[[x]]/\langle F(x, 0) \rangle.$$

Any deformation can be induced from the versal deformation by specifying parameters.

$F = f + \sum t_i m_i$  is a *versal deformation of  $f$*  where  $m_1, \dots, m_\tau$  is basis for

$$\mathbb{C}[[x]]^p / f \mathbb{C}[[x]]^p + \begin{pmatrix} \partial f_1 / \partial x_1 \\ \vdots \\ \partial f_p / \partial x_1 \end{pmatrix} \mathbb{C}[[x]] + \dots + \begin{pmatrix} \partial f_1 / \partial x_n \\ \vdots \\ \partial f_p / \partial x_n \end{pmatrix} \mathbb{C}[[x]].$$

## 2 Characterization of normal form of 2-jet of singularities

Let  $I = \langle f, g \rangle \subseteq \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$  defines a complete intersection singularity and  $I_2$  be the 2-jet of  $I$ . According to C.T.C. Wall's classification the 2-jet of  $\langle f, g \rangle$  is a homogenous ideal generated by 2 polynomials of degree 2. We want to give a description of the type of a singularity without producing the normal form. C.T.C. Wall's classification is based on the classification of the 2-jet  $I_2$  of  $\langle f, g \rangle$ . Let  $\bigcap_{i=1}^s Q_i$  be the irredundant primary decomposition of  $I_2$  in  $\mathbb{C}[[x, y, z]]$ . Let  $d_i = \dim_{\mathbb{C}}(\mathbb{C}[[x, y, z]]/Q_i)$ ,  $i = 1, \dots, s$  and  $h_i$  be the Hilbert polynomial of  $\mathbb{C}[[x, y, z]]/Q_i$ . According to C.T.C. Wall's classification we obtain unimodular singularities only in the following cases.

Table 1:

Type	Characterization	Normal form of $I_2$
$P$	$s = 2, d_1 = d_2 = 1 \quad h_1 = h_2 = 2$	$\langle x^2, yz \rangle$
$J$	$s = 2, d_1 = d_2 = 1 \quad h_1 = 1, h_2 = 4$	$\langle xy + z^2, xz \rangle$
$F$	$s = 2, d_1 = 1, d_2 = 2 \quad h_1 = 1, h_2 = 1 + t$	$\langle xy, xz \rangle$
$H$	$s = 2, d_1 = 1, d_2 = 2 \quad h_1 = 2, h_2 = 1 + t$	$\langle xy, x^2 \rangle$
$G$	$s = 1$ and $\sqrt{I_2}^3 \subseteq I_2$	$\langle x^2, y^2 \rangle$
$K$	$s = 1$ and $\sqrt{I_2}^3 \not\subseteq I_2$	$\langle xy + z^2, x^2 \rangle$

### 3 Unimodular complete intersection space curve singularities

We set

$$l_i(x, y) = \begin{cases} xy^q, & \text{if } i = 2q \\ y^{q+2}, & \text{if } i = 2q + 1 \end{cases}$$

for brevity.

Assume the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy, z^2 \rangle$ . In this case according to C.T.C. Wall's classification the unimodular space curves are given in the table below

Table 2:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$P_{k,l}$	$\langle xy, x^k + y^l + z^2 \rangle$ $k \geq l \geq 3, k > 3$	$l + k + 1$	$l + k + 1$	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$ $l, k$ even $\langle 1 \rangle, \langle 1 \rangle, \langle 2, k \rangle$ $l$ is even, $k$ is odd $\langle 2, k \rangle, \langle 2, l \rangle$ $l, k$ odd

**Proposition 3.1.** *Let  $I = \langle f, g \rangle \subseteq \mathbb{C}[[x, y, z]]$  defines an isolated complete intersection singularity  $(V(I), 0) \subseteq (\mathbb{C}^3, 0)$ . Let  $\mu$  be the Milnor number of  $I$ . Assume that the 2-jet of  $I$  has normal form  $\langle x^2, yz \rangle$ .*

*If  $(V(I), 0)$  has 4 branches and all branches are smooth, let  $J_1, J_2, J_3$  be the ideals of the strict transform of  $I$  blowing up  $\mathbb{C}^3$  at 0 corresponding to the*

three affine charts. Assume  $(V(J_1), 0)$  is an  $A_{l-3}$  singularity and  $(V(J_2), 0)$  is an  $A_{k-3}$  singularity. Then  $I$  is unimodular of type  $P_{k,l}$ ,  $k \geq 4$  and  $l \geq 3$ .

If  $(V(I), 0)$  has 3 branches, two branches are smooth and the third branch has a semigroup generated by  $(2, k)$  then  $I$  is unimodular of type  $P_{k, \mu-k-1}$  if  $(k, \mu) \neq (4, 8)$  and  $\mu - k > 3$ .

If  $(V(I), 0)$  has 2 branches and the semigroup of the two branches are  $(2, k)$  and  $(2, l)$  then  $I$  is unimodular of type  $P_{k,l}$  if  $(k, l) \neq (3, 3)$  and  $(k, l) \neq (5, 3)$ .

*Proof.* Using lemma 3.2 (cf. [ADPG1]) we may assume  $I = \langle x^2 + y^k + z^l + g, yz + h \rangle$ ,  $3 \leq k \leq l \leq \infty, g \in \langle x, y, z \rangle^{l+1}, h \in \langle x, y, z \rangle^3$ . According to Wall's classification we may assume that  $I = \langle yz, x^2 + z^l + y^k \rangle$ . Then  $I = \langle z, x^2 + y^k \rangle \cap \langle y, x^2 + z^l \rangle$ . If  $l$  and  $k$  are even then  $(V(I), 0)$  has 4 smooth branches. If  $l + k$  is odd then  $(V(I), 0)$  has 3 branches, 2 of them smooth and the third defining an  $A_{k-1}$  respectively  $A_{l-1}$  singularity. If  $l$  and  $k$  are odd then  $(V(I), 0)$  has 2 branches, an  $A_{k-1}$  respectively  $A_{l-1}$  singularity. This proves the second and third part of the Proposition. For the first part we have to identify  $k$  and  $l$ . To do this we blow up of 0 of  $\mathbb{C}^3$  and consider the strict transform in the 3 affine charts. We obtain  $J_1 = \langle y, x^2 + z^{l-2} \rangle$ ,  $J_2 = \langle z, x^2 + y^{k-2} \rangle$ ,  $J_3 = \langle yz, 1 + z^l x^{l-2} + y^k x^{k-2} \rangle$ .  $(V(J_1), 0)$  is an  $A_{l-3}$  singularity and  $(V(J_2), 0)$  is an  $A_{k-3}$  singularity.  $\square$

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**Algorithm 1** Psingularity(I)

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**Input:**  $I = \langle f, g \rangle \in \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$  and 2-jet of  $I$   
having normal form  $(xy, z^2)$

**Output:** the type of the singularity

- 1: compute  $\mu =$  Milnor number of the  $I$ ;
  - 2: compute  $\tau =$  Tjurina number of the  $I$ ;
  - 3: compute  $B =$  semigroups of  $I$  corresponding to the branches;
  - 4:  $T = \text{findlk}(I)$ ; \*
  - 5: **if**  $\mu = \tau$  and  $\mu = T[1] + T[2] + 1$  **then**
  - 6:     **if**  $T[1]$  and  $T[2]$  even and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$  **then**
  - 7:         **return**  $(P_{T[1], T[2]})$ ;
  - 8:     **if**  $T[1] + T[2]$  odd and  $B = \langle 2, T[1] \rangle, \langle 1 \rangle, \langle 1 \rangle$  or  $B = \langle 2, T[2] \rangle, \langle 1 \rangle, \langle 1 \rangle$  **then**
  - 9:         **return**  $(P_{T[1], T[2]})$ ;
  - 10:     **if**  $T[1]$  and  $T[2]$  odd and  $B = (2, T[1]), (2, T[2])$  **then**
  - 11:         **return**  $(P_{T[1], T[2]})$ ;
  - 12: **return** (not unimodular);
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\***findlk**(I) is a procedure which computes  $k$  and  $l$  for the given  $I = \langle xy, x^2 + y^l + z^k \rangle$ .

Assume the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy, xz \rangle$ . According to C.T.C. Wall's classification all unimodular curve singularities are in the  $\mu$ -constant strata of the versal deformation of the curve singularities given in the Table 3.

Table 3:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$FT_{4,4}$	$\langle xy + z^3, xz + y^3 + \lambda yz^2 \rangle$	10	10	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$
$FT_{k,l}$	$\langle xy + z^{l-1}, xz + y^{k-1} + yz^2 \rangle$ $k \geq l \geq 4, k > 4$	$l + k + 2$	$l + k + 1$	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$ if $k, l$ even $\langle 1 \rangle, \langle 2, l - 2 \rangle, \langle 2, k - 2 \rangle$ if $k, l$ odd $\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, k - 2 \rangle$ if $k$ odd, $l$ even
$FW_{13}$	$\langle xy + z^3, xz + y^4 \rangle$	13	13	$\langle 1 \rangle, \langle 4, 5, 11 \rangle$
$FW_{14}$	$\langle xy + z^3, xz + zy^3 \rangle$	14	14	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 4 \rangle$
$FW_{1,0}$	$\langle xy + z^3, xz + z^2y^2 + \lambda y^5 \rangle$ $\lambda \neq 0, -1/4$	16	16	$\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$
$FW_{1,i}$	$\langle xy + z^3, xz + z^2y^2 + y^{5+i} \rangle$	$16 + i$	$16 + i - 2$	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, 3 \rangle$ if $i$ odd $\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, \mu - 13 \rangle$ if $i$ even
$FW'_{1,i}$	$\langle xy + z^3, xz + 2z^2y^2 - y^5 + zy^2l_i(z, y) \rangle$	$16 + i$	$14 + i - 2$	$\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$  if $i$ even $\langle 1 \rangle, \langle 4, 6, \tau - 2, \tau \rangle$ if $i$ odd
$FW_{18}$	$\langle xy + z^3, xz + zy^4 \rangle$	18	18	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 5 \rangle$
$FW_{19}$	$\langle xy + z^3, xz + y^6 \rangle$	19	19	$\langle 1 \rangle, \langle 4, 7, 17 \rangle$
$FZ_{6m+6}$	$\langle xy, xz + z^3 + y^{3m+1} \rangle$	$6m + 6$	$6m + 6$	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 3m + 1 \rangle$
$FZ_{6m+7}$	$\langle xy, xz + z^3 + zy^{2m+1} \rangle$	$6m + 7$	$6m + 7$	$\langle 1 \rangle, \langle 1 \rangle, \langle 2, 2m + 1 \rangle$
$FZ_{6m+8}$	$\langle xy, xz + z^3 + y^{3m+2} \rangle$	$6m + 8$	$6m + 8$	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 3m + 2 \rangle$
$FZ_{m-1,0}$	$\langle xy, xz + z^3 + z^2y^m + \lambda y^{3m} \rangle$ $\lambda \neq 0, -4/27$	$6m + 4$	$6m + 4$	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, 2m + i \rangle$
$FZ_{m-1,i}$	$\langle xy, xz + z^3 + z^2y^m + y^{3m+i} \rangle$	$6m + 4 + i$	$5m + 4 + i$	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, 2m + i \rangle$ if $i$ odd $\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$ if $i$ even

**Proposition 3.2.** *The unimodular complete intersection curve singularities with Milnor number 13, 2 branches and semigroup  $\langle 1 \rangle, \langle 4, 5, 11 \rangle$  are  $FW_{13}$  with Tjurina number 13 defined by the ideal  $\langle xy + z^3, xz + y^4 \rangle$  and  $FW_{13,1}$  with Tjurina number 12 defined by the ideal  $\langle xy + z^3, xz + y^4 + y^2z^2 \rangle$ .*

*Proof.* In the list of C.T.C. Wall  $FW_{13}$  defined by the ideal  $\langle xy + z^3, xz + y^4 \rangle$  is the only singularity with  $\mu = 13$ , 2 branches and semigroup  $\langle 1 \rangle, \langle 4, 5, 11 \rangle$ .

The versal deformation of  $FW_{13}$  is given by  $\langle xy + z^3 + \nu_1 z^2 + \nu_2 z + \nu_3, xz + y^4 + \lambda_1 y^2 z^2 + \lambda_2 y z^2 + \lambda_3 z^2 + \lambda_4 y^2 z + \lambda_5 y z + \lambda_6 z + \lambda_7 y^3 + \lambda_8 y^2 + \lambda_9 \rangle$ .  $FW_{13}$

defines a weighted homogenous isolated complete intersection singularity with weights  $(w_1, w_2, w_3) = (11, 4, 5)$  and degrees  $(d_1, d_2) = (15, 16)$ . The versal  $\mu$ -constant deformation of  $FW_{13}$  is given by  $\langle xy + z^3, xz + y^4 + \lambda_1 y^2 z^2 \rangle$ .

Using the coordinate change  $x \rightarrow \xi^{11}x, y \rightarrow \xi^4y, z \rightarrow \xi^5z$  we have  $I_\lambda = \langle xy + z^3, xz + y^4 + \xi^2 \lambda_1 y^2 z^2 \rangle$ . Choose  $\xi$  such that  $\xi^2 \lambda_1 = 1$ . So we obtain  $\langle xy + z^3, xz + y^4 + y^2 z^2 \rangle$ . It has 2 branches and same semigroup as  $FW_{13}$  and  $\tau = 12$ .

It can be distinguished from  $FW_{13}$  by the Tjurina number.  $\square$

**Proposition 3.3.** *The unimodular complete intersection curve singularities with Milnor number 14, 3 branches two of them are smooth and the third branch has semigroup  $\langle 3, 4 \rangle$  are  $FW_{14}$  with Tjurina number 14 defined by the ideal  $\langle xy + z^3, xz + zy^3 \rangle$  and  $FW_{14,1}$  with the Tjurina number 13 defined by the ideal  $\langle xy + z^3, xz + zy^3 + y^5 \rangle$ .*

**Proposition 3.4.** *The unimodular complete intersection curve singularities with Milnor number 16, 3 branches and semigroup  $\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$  are  $FW_{1,0}$  with Tjurina number  $\tau = 16$  defined by the ideal  $\langle xy + z^3, xz + z^2 y^2 + \lambda y^5 \rangle$  and  $FW_{1,0,1}$  Tjurina number  $\tau = 15$  defined by the ideal  $\langle xy + z^3, xz + z^2 y^2 + \lambda y^5 + y^6 \rangle$ .*

*Proof.* The proofs of propositions 3.3 and 3.4 are similar to the proof of proposition 3.2.  $\square$

**Proposition 3.5.** *The unimodular complete intersection curve singularities with Milnor number 18, 3 branches and semigroup  $\langle 1 \rangle, \langle 1 \rangle, \langle 3, 5 \rangle$  are  $FW_{18}$  with Tjurina number  $\tau = 18$  defined by the ideal  $\langle xy + z^3, xz + zy^4 \rangle$   $\tau = 18$ ,  $FW_{18,1}$  with Tjurina number  $\tau = 17$  defined by the ideal  $\langle xy + z^3, xz + zy^4 + y^7 \rangle$  and  $FW_{18,2}$  with Tjurina number  $\tau = 16$  defined by the ideal  $\langle xy + z^3, xz + zy^4 + y^6 \rangle$ .*

*Proof.* In the list of C.T.C. Wall  $FW_{18}$  defined by the ideal  $\langle xy + z^3, xz + zy^4 \rangle$  is the only singularity with  $\mu = 18$ , 3 branches and semigroup  $\langle 1 \rangle, \langle 1 \rangle, \langle 3, 5 \rangle$ .

The versal deformation of  $FW_{18}$  is given by  $\langle xy + z^3 + \nu_1 z^2 + \nu_2 z + \nu_3, xz + y^4 z + \lambda_1 y^2 z^2 + \lambda_2 y z^2 + \lambda_3 z^2 + \lambda_4 y^3 z + \lambda_5 y^2 z + \lambda_6 y z + \lambda_7 z + \lambda_8 y^7 + \lambda_9 y^6 + \lambda_{10} y^5 + \lambda_{11} y^4 + \lambda_{12} y^3 + \lambda_{13} y^2 + \lambda_{14} y + \lambda_{15} \rangle$ .  $FW_{18}$  defines a weighted homogenous isolated complete intersection singularity with weights  $(w_1, w_2, w_3) = (12, 3, 5)$  and degrees  $(d_1, d_2) = (15, 17)$ . The versal  $\mu$ -constant deformation of  $FW_{18}$  is given by  $\langle xy + z^3, xz + y^4 z + \lambda_8 y^7 + \lambda_9 y^6 \rangle$ .

If  $\lambda_9 \neq 0$  then we have  $I = \langle xy + z^3, xz + y^4 z + uy^6 \rangle$  where  $u = \lambda_8 y + \lambda_7$ .

Using the coordinate change  $x \rightarrow \xi^{12}x, y \rightarrow \xi^3y, z \rightarrow \xi^5z$  we may assume  $I = \langle xy + z^3, xz + y^4 z + \xi \bar{u} y^6 \rangle$ . Choose  $\xi$  such that  $\xi \bar{u} = 1$ . So  $I = \langle xy + z^3, xz + y^4 z + y^6 \rangle$ . It has Tjurina number  $\tau = 17$ , two branches and the same semigroup as  $FW_{18}$ .

If  $\lambda_9 = 0$  we again apply the same transformation and obtain  $I = \langle xy + z^3, xz + y^4z + y^7 \rangle$  by choosing  $\lambda_8^4 = 1$ . It has Tjurina number  $\tau = 16$ , also 3 branches and the same semigroup as  $FW_{18}$ .  $FW_{18,1}$  and  $FW_{18,2}$  can be differentiated from  $FW_{18}$  by the Tjurina numbers.  $\square$

**Proposition 3.6.** *The unimodular complete intersection curve singularities with Milnor number 19, 2 branches and semigroup  $\langle 1 \rangle, \langle 4, 7, 17 \rangle$  are  $FW_{19}$  with Tjurina number  $\tau = 19$  defined by the ideal  $\langle xy + z^3, xz + y^6 \rangle$ ,  $FW_{19,1}$  with Tjurina number  $\tau = 18$  defined by the ideal  $\langle xy + z^3, xz + y^6 + y^4z^2 \rangle$  and  $FW_{19,2}$  with Tjurina number  $\tau = 17$  defined by the ideal  $\langle xy + z^3, xz + y^6 + y^3z^2 \rangle$ .*

*Proof.* The proof can be done similarly to the proof of proposition 3.5.  $\square$

**Proposition 3.7.** *The unimodular complete intersection singularities having Milnor number of the form  $\mu = 6m + 6$  where  $m$  is a positive integer with 3 branches, 2 branches are smooth and third branch has semigroup  $\langle 3, 3m + 1 \rangle$  are  $FZ_{6m+6}$  defined by the ideal  $\langle xy, xz + z^3 + y^{3m+1} \rangle$  with Tjurina number  $\tau = 6m + 6$  and  $FZ_{6m+6, i+1}$ ,  $i=0, 1, \dots, m-1$ , defined by the ideal  $\langle xy, xz + z^3 + y^{3m+1} + y^{3m-i}z \rangle$  with Tjurina number  $\tau = \mu - i - 1$ .*

*Proof.* In C.T.C. Wall's list  $FZ_{6m+6}$ ,  $m \geq 1$  defined by the ideal  $\langle xy, xz + z^3 + y^{3m+1} \rangle$  are the singularities with  $\mu = 6m + 6$ , 3 branches and semigroup  $\langle 1 \rangle, \langle 1 \rangle, \langle 3, 3m + 1 \rangle$ .

The versal deformation of  $FZ_{6m+6}$  is given by  $\langle xy + \nu_1 z^2 + \nu_2 z + \nu_3, xz + z^3 + y^{3m+1} + \sum_{i=0}^{3m} \alpha_i y^{3m-i} z + \sum_{i=0}^{3m} \beta_i y^{3m-i} \rangle$ .  $FZ_{6m+6}$  defines a weighted homogenous isolated complete intersection singularity with weights  $(w_1, w_2, w_3) = (6m + 2, 3, 3m + 1)$  and degrees  $(d_1, d_2) = (6m + 5, 9m + 3)$ . The versal  $\mu$ -constant deformation of  $FZ_{6m+6}$  is given by  $\langle xy, xz + z^3 + y^{3m+1} + \sum_{i=0}^{m-1} \alpha_i y^{3m-i} z \rangle$ .

Consider  $\phi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y, z]])$  defined by  $\phi(x) = \xi^{6m+2}x$ ,  $\phi(y) = \xi^3y$  and  $\phi(z) = \xi^{3m+1}z$ . If  $\alpha_{m-1} \neq 0$ , then  $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+1}z + \sum_{i=0}^{m-2} \alpha_i y^{(m-1)-i} + \alpha_{m-1} \rangle$ . Let  $u_{m-1} = \sum_{i=0}^{m-2} \alpha_i y^{(m-1)-i} + \alpha_{m-1}$  then  $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+1}z + u_{m-1} \rangle$ . By applying the transformation  $\phi$  we get  $I = \langle xy, xz + z^3 + y^{3m+1} + \xi y^{2m+1}z \bar{u}_{m-1} \rangle$ . Choose  $\xi$  such that  $\xi \bar{u}_{m-1} = 1$ . This implies  $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+1}z \rangle$ .

Now we assume  $\alpha_{m-1} = 0$ . This implies  $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+2}z + \sum_{i=0}^{m-3} \alpha_i y^{(m-2)-i} + \alpha_{m-2} \rangle$ . Let  $u_{m-2} = \sum_{i=0}^{m-3} \alpha_i y^{(m-2)-i} + \alpha_{m-2}$  then  $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+2}z + u_{m-2} \rangle$ . After applying  $\phi$  we may assume that  $I = \langle xy, xz + z^3 + y^{3m+1} + \xi^4 y^{2m+2}z \bar{u}_{m-2} \rangle$ . Choose  $\xi$  such that  $\xi^4 \bar{u}_{m-2} = 1$ .



This implies  $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+2}z \rangle$ . If  $\alpha_{m-2} = 0$  then we assume  $\alpha_{m-3} \neq 0$ .

We may iterate this process and we get  $m$  different unimodular singularities  $I = \langle xy, xz + z^3 + y^{3m+1} + y^{3m-i}z \rangle$ ,  $i = 0, 1, \dots, m-1$  having Tjurina number  $\tau = \mu - i - 1$  and the same semigroup as  $FZ_{6m+6}$ . These singularities can be distinguished from  $FZ_{6m+6}$  by the Tjurina numbers.  $\square$

**Proposition 3.8.** *The unimodular complete intersection singularities having Milnor number of the form  $\mu = 6m + 7$  where  $m$  is a positive integer with 4 branches, 3 branches are smooth and the fourth branch has semigroup generated by  $\langle 2, 2m + 1 \rangle$  are  $FZ_{6m+7}$  with Tjurina number  $\tau = 6m + 7$  defined by the ideal  $\langle xy, xz + z^3 + zy^{2m+1} \rangle$  and  $FZ_{6m+7,i}$ ,  $i = 1, \dots, m$  with Tjurina number  $\tau = \mu - i$  defined by the ideal  $\langle xy, xz + z^3 + zy^{2m+1} + y^{4m+2-i} \rangle$ .*

**Proposition 3.9.** *The unimodular complete intersection singularities having Milnor number of the form  $\mu = 6m + 8$  where  $m$  is a positive integer with 3 branches and the semigroup  $\langle 1, \langle 1 \rangle, \langle 3, 3m + 2 \rangle$  are  $FZ_{6m+8}$  defined by the ideal  $\langle xy, xz + z^3 + y^{3m+2} \rangle$  with Tjurina number  $\tau = 6m + 8$  and  $FZ_{6m+8,i}$ ,  $i = 1, \dots, m$  defined by the ideal  $\langle xy, xz + z^3 + y^{3m+2} + y^{3m+2-i}z \rangle$  with Tjurina number  $\tau = \mu - i$ .*

*Proof.* The proofs of propositions 3.8 and 3.9 are similar to proof of proposition 3.7.  $\square$

Summarizing the results of the above propositions we complete the list of unimodular complete intersection singularities in case of  $\langle f, g \rangle$  having 2-jet with normal form  $\langle xy, xz \rangle$ .

**Proposition 3.10.** *Let  $(V(\langle f, g \rangle), 0) \subseteq (\mathbb{C}^3, 0)$  be the germ of a complete intersection space curve singularity. Assume it is not a hypersurface singularity and the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy, xz \rangle$ .  $(V(\langle f, g \rangle), 0)$  is unimodular if and only if it is isomorphic to a complete intersection in Table 3 and 4.*

*Proof.* The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.2 - 3.9.  $\square$

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**Algorithm 2** Fsingularity(I)

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**Input:**  $I = \langle f, g \rangle \in \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$  and 2-jet of  $I$   
having normal form  $(xy, xz)$ .

**Output:** the type of the singularity

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1: compute  $\mu$  =Milnor number of the  $I$ ;
2: compute  $\tau$  =Tjurina number of the  $I$ ;
3: compute  $B$  =semigroups of  $I$  corresponding to the branches;
4: if  $\mu = 10$  and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$  then
5:   if  $\mu = \tau$  then
6:     return  $(FT_{4,4})$ ;
7: if  $\mu = 13$  and  $B = \langle 1 \rangle, \langle 4, 5, 11 \rangle$  then
8:   if  $\mu = \tau$  then
9:     return  $(FW_{13})$ ;
10:  if  $\mu - \tau = 1$  then
11:    return  $(FW_{13,1})$ ;
12: if  $\mu = 14$  and  $A = \langle 1 \rangle, \langle 1 \rangle, \langle 3, 4 \rangle$  then
13:   if  $\mu = \tau$  then
14:     return  $(FW_{14})$ ;
15:   if  $\mu - \tau = 1$  then
16:     return  $(FW_{14,1})$ ;
17: if  $\mu = 18$  and  $A = \langle 1 \rangle, \langle 1 \rangle, \langle 3, 5 \rangle$  then
18:   if  $\mu = \tau$  then
19:     return  $(FW_{18})$ ;
20:   else
21:     return  $(FW_{18, \mu - \tau})$ ;
22: if  $\mu = 19$  and  $B = \langle 1 \rangle, \langle 4, 7, 17 \rangle$  then
23:   if  $\mu = \tau$  then
24:     return  $(FW_{19})$ ;
25:   else
26:     return  $(FW_{19, \mu - \tau})$ ;
27: if  $\mu = 16$  and  $B = \langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$  then
28:   if  $\mu = \tau$  then
29:     return  $(FW_{1,0})$ ;
30:   if  $\mu - \tau = 1$  then
31:     return  $(FW_{1,0,1})$ ;
32: if  $\mu \equiv 0 \pmod{6}$ ,  $\mu > 11$  and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 3, 3(\mu - 6)/6 + 1 \rangle$  then
33:   if  $\mu = \tau$  then
34:     return  $(FZ_{\mu})$ ;
35:   else
36:     return  $(FZ_{\mu, \mu - \tau})$ ;

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1: if  $\mu \equiv 1 \pmod{6}$ ,  $\mu > 12$  and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 2, 2(\mu - 7)/6 + 1 \rangle$  then
2:   if  $\mu = \tau$  then
3:     return  $(FZ_\mu)$ ;
4:   else
5:     return  $(FZ_{\mu, \mu - \tau})$ ;
6: if  $\mu \equiv 2 \pmod{6}$ ,  $\mu > 13$  and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 3, 3(\mu - 8)/6 + 2 \rangle$  then
7:   if  $\mu = \tau$  then
8:     return  $(FZ_\mu)$ ;
9:   else
10:    return  $(FZ_{\mu, \mu - \tau})$ ;
11: if  $\mu \equiv 4 \pmod{6}$  and  $\mu > 9$  then
12:    $m = (\mu - 4)/6$ ;
13:   if  $\mu = \tau$  then
14:     if  $m$  is even and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$  then
15:       return  $(FZ_{m-1, 0})$ ;
16:     if  $m$  is odd and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 2, 3(\mu - 4)/6 \rangle$  then
17:       return  $(FZ_{m-1, 0})$ ;
18:   if  $\mu \neq \tau$  then
19:     if  $\mu - \tau = 1$  then
20:        $T = \text{findlk}(I)$ ;
21:       if  $\mu = T[1] + T[2] + 2$  then
22:         if  $T[1]$  and  $T[2]$  even and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$  then
23:           return  $(FT_{T[1], T[2]})$ ;
24:         if  $T[1] + T[2]$  odd and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, k - 2 \rangle$  or  $B =$ 
25:            $\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, l - 2 \rangle$  then
26:             return  $(FT_{T[1], T[2]})$ ;
27:           if  $T[1]$  and  $T[2]$  odd and  $B = \langle 1 \rangle, \langle 2, l - 2 \rangle, \langle 2, k - 2 \rangle$  then
28:             return  $(FT_{T[1], T[2]})$ ;
29:       if  $\mu$  is odd and  $\mu > 16$  then
30:         if  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, 3 \rangle$  then
31:           return  $(FW_{1, \mu - 16})$ ;
32:         if  $B = \langle 1 \rangle, \langle 4, 6, \mu - 4, \mu - 2 \rangle$  then
33:           return  $(FW'_{1, \mu - 16})$ ;
34:       if  $\mu$  is even and  $\mu > 16$  then
35:         if  $B = \langle 1 \rangle, \langle 2, 3, \langle 2, \mu - 13 \rangle$  then
36:           return  $(FW_{1, \mu - 16})$ ;
37:         if  $B = \langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$  then
38:           return  $(FW'_{1, \mu - 16})$ ;
39:       if  $\mu$  is even,  $\mu \geq 11$  and  $B = \langle 1, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$  then
40:         return  $(FZ_{\mu - \tau - 1, 6\tau - 5\mu - 4})$ ;
41:       if  $\mu$  is odd,  $\mu \geq 11$  and  $B = \langle 1, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 4\tau - 3\mu - 4, 2 \rangle$  then
42:         return  $(FZ_{\mu - \tau - 1, 6\tau - 5\mu - 4})$ ;

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Table 4:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$FW_{13,1}$	$\langle xy + z^3, xz + y^4 + y^2z^2 \rangle$	13	12	$\langle 1 \rangle, \langle 4, 5, 11 \rangle$
$FW_{14,1}$	$\langle xy + z^3, xz + zy^3 + y^5 \rangle$	14	13	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 4 \rangle$
$FW_{1,0,1}$	$\langle xy + z^3, xz + z^2y^2 + \lambda y^5 + y^6 \rangle$ $\lambda \neq 0, -1/4$	16	15	$\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$
$FW_{18,1}$	$\langle xy + z^3, xz + zy^4 + y^7 \rangle$	18	17	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 5 \rangle$
$FW_{18,2}$	$\langle xy + z^3, xz + zy^4 + y^6 \rangle$	18	16	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 5 \rangle$
$FW_{19,1}$	$\langle xy + z^3, xz + y^6 + y^4z^2 \rangle$	19	18	$\langle 1 \rangle, \langle 4, 7, 17 \rangle$
$FW_{19,2}$	$\langle xy + z^3, xz + y^6 + y^3z^2 \rangle$	19	17	$\langle 1 \rangle, \langle 4, 7, 17 \rangle$
$FZ_{6m+6,i}$	$\langle xy, xz + z^3 + y^{3m+1} + y^{3m-i}z \rangle$ $i = 0, 1, \dots, m-1$	$6m+6$	$6m+5-i$	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 3m+1 \rangle$
$FZ_{6m+7,i}$	$\langle xy, xz + z^3 + zy^{2m+1} + y^{4m+2-i}z \rangle$ $i = 1, \dots, m$	$6m+7$	$6m+7-i$	$\langle 1 \rangle, \langle 1 \rangle, \langle 2, 2m+1 \rangle$
$FZ_{6m+8,i}$	$\langle xy, xz + z^3 + y^{3m+2} + y^{3m+2-i}z \rangle$ $i = 1, \dots, m$	$6m+8$	$6m+8-i$	$\langle 1 \rangle, \langle 1 \rangle, \langle 3, 3m+2 \rangle$

**Proposition 3.11.** *Assume the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle x^2, y^2 \rangle$ . According to C.T.C. Wall's classification the unimodular space curve singularities are given in the table below*

Table 5:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$G_{2n+3} \ n \geq 3$	$(x^2 + z^3, y^2 + z^n)$	$2n+3$	$2n+3$	$(2, 3), (2, 3)$
$G_{2n+6} \ n \geq 2$	$(x^2 + z^3, y^2 + xz^n)$	$2n+6$	$2n+6$	$(4, 6, 2n+3)$

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**Algorithm 3**  $G_{\text{singularity}}(I)$

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**Input:**  $I = \langle f, g \rangle \in \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$  having 2-jet  
of the form  $(x^2, y^2)$ .

**Output:** the type of the singularity

- 1: compute  $\mu$  =Milnor number of  $I$ ;
  - 2: compute  $\tau$  =Tjurina number of the  $I$ ;
  - 3: compute  $B$  =semigroups of  $I$  corresponding to the branches;
  - 4: **if**  $\mu = \tau$  **then**
  - 5:     **if**  $\mu$  is even and  $B = (4, 6, \mu - 3)$  **then**
  - 6:         **return**  $(G_\mu)$ ;
  - 7:     **if**  $\mu$  is odd and  $B = (2, 3), (2, 3)$  **then**
  - 8:         **return**  $(G_\mu)$ ;
  - 9: **return** (*not unimodular*);
- 

Assume the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy, x^2 \rangle$ . According to C.T.C. Wall's classification all unimodular curve singularities are in the  $\mu$ -constant strata of the versal deformation of the curve singularities given in the table below

Table 6:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$HA_{11}$	$\langle xy + z^3, x^2 + z^3 + yz^2 + y^3 \rangle$	11	11	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, 3 \rangle$
$HA_{r+11}$	$\langle xy + z^3, x^2 + z^3 + yz^2 + y^{3+r} \rangle$ $r \geq 1$	$r + 11$	$r + 10$	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, 3 \rangle, \mu$ <i>odd</i> $\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 2 + r \rangle, \mu$ <i>even</i>
$HB_{r+13}$	$\langle xy + z^3, x^2 + yz^2 + y^{4+r} \rangle$ $r \geq 0$	$r + 13$	$r + 12$	$\langle 3, 4, 5 \rangle, \langle 2, 3 + r \rangle$
$HC_{13}$	$\langle xy + z^3, x^2 + z^3 + y^4 \rangle$	13	13	$\langle 2, 3 \rangle, \langle 3, 4 \rangle$
$HC_{14}$	$\langle xy + z^3, x^2 + z^3 + zy^3 \rangle$	14	14	$\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$
$HC_{15}$	$\langle xy + z^3, x^2 + z^3 + y^5 \rangle$	15	15	$\langle 2, 3 \rangle, \langle 3, 5 \rangle$
$HD_{13}$	$\langle xy + z^3, x^2 + zy^2 \rangle$	13	13	$\langle 1 \rangle, \langle 4, 5, 7 \rangle$
$HD_{14}$	$\langle xy + z^3, x^2 + y^3 \rangle$	14	14	$\langle 5, 6, 9 \rangle$

**Proposition 3.12.** *The unimodular complete intersection curve singularities with Milnor number 13, 2 branches and with semigroups  $\langle 2, 3 \rangle, \langle 3, 4 \rangle$  are  $HC_{13}$  with Tjurina number 13 defined by the ideal  $\langle xy + z^3, x^2 + z^3 + y^4 \rangle$  and  $HC_{13,1}$  with Tjurina number  $\tau = 12$  defined by the ideal  $\langle xy + z^3, x^2 + z^3 + y^4 + y^3z \rangle$ .*

The singularities having the same Milnor number but being irreducible with semigroup  $\langle 5, 6, 9 \rangle$  are  $HD_{13}$  defined by the ideal  $\langle xy + z^3, x^2 + zy^2 \rangle$  with Tjurina number 13 and  $HD_{13,1}$  with Tjurina number  $\tau = 12$  defined by the ideal  $\langle xy + z^3 + yz^2, x^2 + zy^2 + zy^{13} \rangle$ .

*Proof.* In the list of C.T.C. Wall  $HC_{13}$  defined by the ideal  $I = \langle xy + z^3, x^2 + z^3 + y^4 \rangle$  is the only singularity with  $\mu = 13$ , 2 branches and semigroup  $\langle 2, 3 \rangle, \langle 3, 4 \rangle$ .

We may choose an automorphism  $\psi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y, z]])$  such that  $\psi(I) = \langle xy, x^2 + z^3 + y^4 \rangle$ . The versal deformation is given by  $\langle xy + \nu_1 z^2 + \nu_2 yz + \nu_3 z + \nu_4 y + \nu_5, x^2 + z^3 + y^4 + \lambda_1 y^3 z + \lambda_2 y^2 z + \lambda_3 yz + \lambda_4 z + \lambda_5 y^3 + \lambda_6 y^2 + \lambda_7 y + \lambda_8 \rangle$ .  $HC_{13}$  defines a weighted homogenous isolated complete intersection singularity with weights  $\langle w_1, w_2, w_3 \rangle = \langle 6, 3, 4 \rangle$  and degrees  $\langle d_1, d_2 \rangle = \langle 9, 12 \rangle$ . The versal  $\mu$ -constant deformation of  $HC_{13}$  is given by  $\langle xy, x^2 + z^3 + y^4 + \lambda_1 y^3 z \rangle$ . Using the coordinate change  $x \rightarrow \xi^6 x, y \rightarrow \xi^3 y, z \rightarrow \xi^4 z$ , we may assume  $I_{\lambda_1} = \langle xy + z^3, x^2 + z^3 + y^4 + \xi \lambda_1 y^3 z \rangle$ . Choose  $\xi$  such that  $\xi \lambda_1 = 1$ . So  $\langle xy, x^2 + z^3 + y^4 + y^3 z \rangle$  has two branches with same semigroup as  $HC_{13}$  and Tjurina number  $\tau = 12$ . It can be differentiated from  $HC_{13}$  by the Tjurina number.

In C.T.C. Wall's list  $HD_{13}$  defined by the ideal  $\langle xy + z^3, x^2 + zy^2 \rangle$  is the only singularity with  $\mu = 13$ , 2 branches and semigroup  $\langle 1 \rangle, \langle 4, 5, 7 \rangle$ .

The versal deformation of  $HD_{13}$  is given by  $\langle xy + z^3 + \nu_1 yz^2 + \nu_2 z^2 + \nu_3 yz + \nu_4 z + \nu_5 y + \nu_6, x^2 + zy^2 + \lambda_1 z^3 + \lambda_2 yz^2 + \lambda_3 z^2 + \lambda_4 yz + \lambda_5 z + \lambda_6 y + \lambda_7 \rangle$ .  $HD_{13}$  defines a weighted homogenous isolated complete intersection singularity with weights  $\langle w_1, w_2, w_3 \rangle = \langle 7, 5, 4 \rangle$  and degrees  $\langle d_1, d_2 \rangle = \langle 12, 14 \rangle$ . The versal  $\mu$ -constant deformation of  $HD_{13}$  is given by  $\langle xy + z^3 + \nu_1 yz^2, x^2 + y^2 z \rangle$ .

Using the coordinate change  $x \rightarrow \xi^7 x, y \rightarrow \xi^5 y, z \rightarrow \xi^4 z$ , we may assume  $I_{\lambda} = \langle xy + z^3 + \xi \nu_1 yz^2, x^2 + y^2 z \rangle$ . Choose  $\xi$  such that  $\xi \nu_1 = 1$ . So  $\langle xy + z^3 + yz^2, x^2 + y^2 z \rangle$  has 2 branches with same semigroup as  $HD_{13}$  and Tjurina number  $\tau = 12$ . It can be differentiated from  $HD_{13}$  by the Tjurina number.  $\square$

**Proposition 3.13.** *The unimodular complete intersection curve singularities with Milnor number 14, 3 branches and with semigroup  $\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$  are  $HC_{14}$  with Tjurina number 14 defined by the ideal  $\langle xy + z^3, x^2 + z^3 + zy^3 \rangle$  and  $HC_{14,1}$  with Tjurina number  $\tau = 13$  defined by the ideal  $\langle xy + z^3, x^2 + z^3 + zy^3 + y^5 \rangle$ . The singularities having same Milnor number but irreducible with semigroup  $\langle 5, 6, 9 \rangle$  are  $HD_{14}$  defined by the ideal  $\langle xy + z^3, x^2 + y^3 \rangle$  with Tjurina number 13 and  $HD_{14,1}$  with Tjurina number  $\tau = 13$  defined by the ideal  $\langle xy + z^3, x^2 + y^3 + z^4 \rangle$ .*

**Proposition 3.14.** *The unimodular complete intersection curve singularities with Milnor number 15, 2 branches and semigroup  $\langle 2, 3 \rangle, \langle 3, 5 \rangle$  are  $HC_{15}$  with*

*Tjurina number 15 defined by the ideal  $\langle xy + z^3, x^2 + z^3 + y^5 \rangle$  and  $HC_{15,1}$  with Tjurina number  $\tau = 14$  defined by the ideal  $\langle xy + z^3, x^2 + z^3 + y^5 + y^4z \rangle$ .*

*Proof.* The proofs of Propositions 3.13 and 3.14 are similar to proof of Proposition 3.12.  $\square$

Summarizing the results of the propositions above we complete the list of unimodular complete intersection singularities in case of  $\langle f, g \rangle$  having 2-jet with normal form  $\langle xy, x^2 \rangle$ .

Table 7:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$HC_{13,1}$	$\langle xy + z^3, x^2 + z^3 + y^4 + y^3z \rangle$	13	12	$\langle 2, 3 \rangle, \langle 3, 4 \rangle$
$HC_{14,1}$	$\langle xy + z^3, x^2 + z^3 + zy^3 + y^5 \rangle$	14	13	$\langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$
$HC_{15,1}$	$\langle xy + z^3, x^2 + z^3 + y^5 + y^4z \rangle$	15	14	$\langle 2, 3 \rangle, \langle 3, 5 \rangle$
$HD_{13,1}$	$\langle xy + z^3 + yz^2, x^2 + zy^2 \rangle$	13	12	$\langle 1 \rangle, \langle 4, 5, 7 \rangle$
$HD_{14,1}$	$\langle xy + z^3, x^2 + y^3 + z^4 \rangle$	14	13	$\langle 5, 6, 9 \rangle$

**Proposition 3.15.** *Let  $(V(\langle f, g \rangle), 0) \subseteq (\mathbb{C}^3, 0)$  be the germ of a complete intersection space curve singularity. Assume it is not a hypersurface singularity and the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy, x^2 \rangle$ .  $(V(\langle f, g \rangle), 0)$  is unimodular if and only if it is isomorphic to a complete intersection in the Table 6 and 7.*

*Proof.* The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.12 - 3.14.  $\square$

---

**Algorithm 4** Hsingularity(I)

---

**Input:**  $I = \langle f, g \rangle \in \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$  2-jet of  $I$  having normal form  $(xy, x^2)$

**Output:** the type of the singularity

- 1: compute  $\mu$  =Milnor number of the  $I$  and  $\tau$  =Tjurina number of the  $I$ ;
  - 2: compute  $B$  =semigroups of  $I$  corresponding to the branches;
  - 3: **if**  $\mu = 13$  **then**
  - 4:   **if**  $\mu = \tau$  **then**
  - 5:     **if**  $B = \langle 2, 3 \rangle, \langle 3, 4 \rangle$  **then**
  - 6:       **return**  $(HC_{13})$ ;
  - 7:     **if**  $B = \langle 1 \rangle, \langle 4, 5, 7 \rangle$  **then**
  - 8:       **return**  $(HD_{13})$ ;
  - 9:   **else**
  - 10:    **if**  $B = \langle 2, 3 \rangle, \langle 3, 4 \rangle$  **then**
  - 11:      **return**  $(HC_{13, \mu - \tau})$ ;
  - 12:    **if**  $B = \langle 1 \rangle, \langle 4, 5, 7 \rangle$  **then**
  - 13:      **return**  $(HD_{13, \mu - \tau})$ ;
  - 14: **if**  $\mu = 14$  **then**
  - 15:   **if**  $\mu = \tau$  **then**
  - 16:     **if**  $B = \langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$  **then**
  - 17:       **return**  $(HC_{14})$ ;
  - 18:     **if**  $B = \langle 5, 6, 9 \rangle$  **then**
  - 19:       **return**  $(HD_{14})$ ;
  - 20:   **else**
  - 21:     **if**  $B = \langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, 3 \rangle$  **then**
  - 22:       **return**  $(HC_{14, \mu - \tau})$ ;
  - 23:     **if**  $B = \langle 5, 6, 9 \rangle$  **then**
  - 24:       **return**  $(HD_{14, \mu - \tau})$ ;
  - 25: **if**  $\mu = 15$  and  $B = \langle 2, 3 \rangle, \langle 3, 5 \rangle$  **then**
  - 26:   **if**  $\mu = \tau$  **then**
  - 27:     **return**  $(HC_{15})$ ;
  - 28:   **else**
  - 29:     **return**  $(HC_{15, \mu - \tau})$ ;
  - 30: **if**  $\mu = 11$  and  $A = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, 3 \rangle$  **then**
  - 31:   **if**  $\mu = \tau$  **then**
  - 32:     **return**  $(HA_{11})$ ;
  - 33: **if**  $\mu \neq \tau$  **then**
  - 34:   **if**  $\mu - \tau = 1$  and  $\mu > 11$  **then**
  - 35:     **if**  $\mu$  is even and  $B = \langle 1 \rangle, \langle 2, 3 \rangle, \langle 2, \mu - 9 \rangle$  **then**
  - 36:       **return**  $(HA_{\mu})$ ;
  - 37:     **if**  $\mu$  is odd and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 2, 3 \rangle$  **then**
  - 38:       **return**  $(HA_{\mu})$ ;
  - 39:     **if**  $\mu$  is even and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 3, 4, 5 \rangle$  **then**
  - 40:       **return**  $(HB_{\mu})$ ;
  - 41:     **if**  $\mu$  is odd and  $B = \langle 3, 4, 5 \rangle, \langle 2, \mu - 10 \rangle$  **then**
  - 42:       **return**  $(HB_{\mu})$ ;
  - 43: **return** *not unimodular*;
-



Assume the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy + z^2, xz \rangle$ . According to C.T.C. Wall's classification all unimodular curve singularities are in the  $\mu$ -constant strata of the versal deformation of the curve singularities given in the table below

Table 8:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$J_{6m+7}$	$\langle xy + z^2, xz + y^{3m+3} \rangle$ $\lambda \neq 0, -4/27$	$6m + 7$	$6m + 7$	$\langle 1 \rangle, \langle 3, 3m + 4, 6m + 5 \rangle$
$J_{6m+8}$	$\langle xy + z^2, xz + zy^{2m+2} \rangle$	$6m + 8$	$6m + 8$	$\langle 1 \rangle, \langle 1 \rangle, \langle 2, 2m + 3 \rangle$
$J_{6m+9}$	$\langle xy + z^2, xz + y^{3m+4} \rangle$	$6m + 9$	$6m + 9$	$\langle 1 \rangle, \langle 3m + 5, 6m + 7 \rangle$
$J_{m+1,0}$	$\langle xy + z^2, xz + z2y^m + \lambda y^{3m+2} \rangle$	$6m + 5$	$6m + 5$	$\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$
$J_{m+1,i}$	$\langle xy + z^2, xz + z2y^m + y^{3m+2+i} \rangle$	$6m + i + 5$	$5m + i + 5$	$\langle 1 \rangle, \langle 1 \rangle, \langle 2, 2m + i + 2 \rangle,$ if $i$ is odd $\langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$ if $i$ is even

**Proposition 3.16.** *The unimodular complete intersection singularities having Milnor number of the form  $\mu = 6m + 7$  where  $m$  is a positive integer with 2 branches and semigroup generated by  $\langle 1 \rangle, \langle 3, 3m + 4, 6m + 5 \rangle$  are  $J_{6m+7}$  defined by the ideal  $\langle xy + z^2, xz + y^{3m+3} \rangle$  with Tjurina number  $\tau = 6m + 7$  and  $J_{6m+7,i}$ ,  $i = 0, \dots, m - 1$  defined by the ideal  $\langle xy + z^2, xz + y^{3m+3} + y^{(3m+1)-i}z \rangle$  with Tjurina number  $\tau = \mu - i$ .*

*Proof.* In C.T.C. Wall's list  $J_{6m+7}$ ,  $m \geq 1$  defined by  $\langle xy + z^2, xz + y^{3m+3} \rangle$  are the singularities with  $\mu = 6m + 7$ , 2 branches and semigroup  $\langle 1 \rangle, \langle 3, 3m + 4, 6m + 5 \rangle$ .

The versal deformation of  $J_{6m+7}$  is given by  $\langle xy + z^2 + \nu_1 z + \nu_2, xz + y^{3m+3} + \sum_{i=0}^{3m+1} \alpha_i y^{(3m+1)-i} z + \sum_{i=0}^{3m+2} \beta_i y^{(3m+2)-i} \rangle$ .  $J_{6m+7}$  defines a weighted homogenous isolated complete intersection singularity with weights  $(w_1, w_2, w_3) = (6m + 5, 3, 3m + 4)$  and degrees  $(d_1, d_2) = (6m + 8, 9m + 9)$ . The versal  $\mu$ -constant deformation of  $J_{6m+7}$  is given by  $\langle xy + z^2, xz + y^{3m+3} + \sum_{i=0}^{m-1} \alpha_i y^{(3m+1)-i} z \rangle$ .

Consider  $\phi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y, z]])$  defined by  $\phi(x) = \xi^{6m+5}x$ ,  $\phi(y) = \xi^3y$  and  $\phi(z) = \xi^{3m+4}z$ . If  $\alpha_{m-1} \neq 0$  then  $I = \langle xy + z^2, xz + y^{3m+3} + y^{2m+2}z + \sum_{i=0}^{m-2} \alpha_i y^{(m-1)-i} + \alpha_{m-1} \rangle$ .

Let  $u_{m-1} = \sum_{i=0}^{m-2} \alpha_i y^{(m-1)-i} + \alpha_{m-1}$ . Then  $I = \langle xy + z^2, xz + y^{3m+3} + y^{2m+2}zu \rangle$ . By applying the transformation  $\phi$  we get  $I = \langle xy + z^2, xz + y^{3m+3} + \xi y^{2m+2}z\bar{u}_{m-1} \rangle$ . Choose  $\xi$  such that  $\xi\bar{u}_{m-1} = 1$ . This implies  $I = \langle xy + z^2, xz + y^{3m+3} + y^{2m+2}z \rangle$ .

If  $\alpha_{m-1} = 0$  then  $I = \langle xy + z^2, xz + y^{3m+3} + y^{2m+3}z(\sum_{i=0}^{m-3} \alpha_i y^{(m-2)-i} + \alpha_{m-2}) \rangle$ . Let  $u_{m-2} = \sum_{i=0}^{m-3} \alpha_i y^{(m-2)-i} + \alpha_{m-2}$  then  $I = \langle xy, xz + z^3 + y^{3m+1} + y^{2m+3}zu_{m-2} \rangle$ . After applying  $\phi$  we may assume that  $I = \langle xy + z^2, xz + y^{3m+3} + \xi^4 y^{2m+3}z\bar{u}_{m-2} \rangle$ . Choose  $\xi$  such that  $\xi^4\bar{u}_{m-2} = 1$ . This implies  $I = \langle xy + z^2, xz + y^{3m+3} + y^{2m+3}z \rangle$ .

If  $\alpha_{m-2} = 0$ . We may iterate this process and we get  $m$  different unimodular space curve singularities  $I = \langle xy + z^2, xz + y^{3m+3} + y^{(3m+1)-i}z \rangle$ ,  $i = 0, 1, \dots, m-1$  having Tjurina number  $\tau = \mu - i$ . These singularities can be distinguished from  $J_{6m+7}$  by the Tjurina numbers.  $\square$

**Proposition 3.17.** *The unimodular complete intersection singularities having Milnor number of the form  $\mu = 6m + 8$  where  $m$  is a positive integer with 3 branches and semigroup generated by  $\langle 1 \rangle, \langle 1 \rangle, \langle 2, 2m+3 \rangle$  are  $J_{6m+8}$  with Tjurina number  $\tau = 6m + 8$  defined by the ideal  $\langle xy + z^2, xz + zy^{2m+2} \rangle$  and  $J_{6m+8,i}$ ,  $i = 1, \dots, m$  with Tjurina number  $\tau = \mu - i$  defined by the ideal  $\langle xy + z^2, xz + zy^{2m+2} + y^{4m+4-i} \rangle$ .*

**Proposition 3.18.** *The unimodular complete intersection singularities having Milnor number of the form  $\mu = 6m + 9$  where  $m$  is a positive integer with 2 branches and semigroup generated by  $\langle 1 \rangle, \langle 3m+5, 6m+7 \rangle$  are  $J_{6m+9}$  with Tjurina number  $\tau = 6m + 9$  defined by the ideal  $\langle xy + z^2, xz + y^{3m+4} \rangle$  and  $J_{6m+9,i}$ ,  $i = 1, \dots, m$  with Tjurina number  $\tau = \mu - i$  defined by the ideal  $\langle xy + z^2, xz + y^{3m+4} + y^{(3m+3)-i} \rangle$ .*

*Proof.* The proofs of Propositions 3.17 and 3.18 can be done similarly to the proof of Proposition 3.16.  $\square$

We complete the list of unimodular singularities in this case as

**Proposition 3.19.** *Let  $(V(\langle f, g \rangle), 0) \subseteq (\mathbb{C}^3, 0)$  be the germ of a complete intersection space curve singularity. Assume it is not a hypersurface singularity and the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy + z^2, xz \rangle$ .  $(V(\langle f, g \rangle), 0)$  is unimodular if and only if it is isomorphic to a complete intersection in Table 8 and 9.*

*Proof.* The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.16 - 3.18.  $\square$

Table 9:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$J_{6m+7,i}$	$\langle xy + z^2, xz + y^{3m+3} + y^{(3m+1)-i}z \rangle$	$6m + 7$	$6m + 7 - i$	$\langle 1, \langle 3, 3m + 4, 6m + 5 \rangle \rangle$
$J_{6m+8,i}$	$\langle xy + z^2, xz + zy^{2m+2} + y^{(4m+4)-i} \rangle$	$6m + 8$	$6m + 8 - i$	$\langle 1, \langle 1, \langle 2, 2m + 3 \rangle \rangle \rangle$
$J_{6m+9,i}$	$\langle xy + z^2, xz + y^{3m+4} + y^{(3m+3)-i} \rangle$	$6m + 9$	$6m + 9 - i$	$\langle 1, \langle 3m + 5, 6m + 7 \rangle \rangle$

Assume the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy + z^2, x^2 \rangle$ . According to C.T.C. Wall's classification all unimodular curve singularities are in the  $\mu$ -constant strata of the versal deformation of the curve singularities given in the table below

Table 10:

Type	Normal Form	$\mu$	$\tau$	Semigroup
$K_{1,0}$	$\langle xy + z^2, x^2 + z^2y + \lambda y^4 \rangle$ $\lambda \neq 0, 1/4$	11	11	$\langle 2, 3 \rangle, \langle 2, 3 \rangle$
$K_{1,i}$	$\langle xy + z^2, x^2 + z^2y + y^4 + y^{4+i} \rangle$	$11 + i$	$11 + i - 1$	$\langle 1, \langle 1, \langle 2, 3 \rangle \rangle$ , if $i$ is odd $\langle 2, 3 \rangle, \langle 2, 3 + i \rangle$ , if $i$ is even
$K'_{1,i}$	$\langle xy + z^2, x^2 + 2z^2y + y^4 + zyl_i \langle z, y \rangle \rangle$	$11 + i$	$11 + i - 1$	$\langle 2, 3 \rangle, \langle 2, 3 \rangle$ , if $i$ is odd $\langle 2, 3 \rangle, \langle 2, 8 + i \rangle$ , if $i$ is even
$K_{13}$	$\langle xy + z^2, x^2 + zy^3 \rangle$	13	13	$\langle 1, \langle 3, 5, 7 \rangle \rangle$
$K_{14}$	$\langle xy + z^2, x^2 + y^5 \rangle$	14	14	$\langle 4, 7, 10 \rangle$

**Proposition 3.20.** *The unimodular complete intersection curve singularities with Milnor number 13, 2 branches and semigroup  $\langle 1, \langle 3, 5, 7 \rangle \rangle$  are  $K_{13}$  with Tjurina number 13 defined by the ideal  $\langle xy + z^2, x^2 + zy^3 \rangle$  and  $K_{13,1}$  with Tjurina number  $\tau = 12$  defined by the ideal  $\langle xy + z^2, x^2 + zy^3 + y^5 \rangle$ .*

*Proof.* In the list of C.T.C. Wall  $K_{13}$  defined by the ideal  $\langle xy + z^2, x^2 + zy^3 \rangle$  is the only singularity with  $\mu = 13$ , 2 branches and semigroup  $\langle 1, \langle 3, 5, 7 \rangle \rangle$ .

The versal deformation of  $K_{13}$  is given by  $\langle xy + z^2 + \nu_1z + \nu_2, x^2 + zy^3 + \lambda_1yz^2 + \lambda_2z^2 + \lambda_3y^2z + \lambda_4yz + \lambda_5z + \lambda_6y^5 + \lambda_7y^4 + \lambda_8y^3 + \lambda_9y^2 + \lambda_{10}y + \lambda_{11} \rangle$ .  $K_{13}$  defines a weighted homogenous isolated complete intersection singularity with weights  $(w_1, w_2, w_3) = (7, 3, 5)$  and degrees  $(d_1, d_2) = (10, 14)$ . The versal  $\mu$ -constant deformation of  $K_{13}$  is given by  $\langle xy + z^2, x^2 + zy^3 + \lambda_6y^5 \rangle$ .

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**Algorithm 5** Jsingularity(I)

---

**Input:**  $I = \langle f, g \rangle \in \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$  and 2-jet of  $I$   
having normal form  $(xy + z^2, xz)$

**Output:** the type of the singularity

- 1: compute  $\mu$  =Milnor number of the  $I$ ;
  - 2: compute  $\tau$  =Tjurina number of the  $I$ ;
  - 3: compute  $B$  =Semigroup of  $I$  corresponding to each branch.;
  - 4: **if**  $\mu \equiv 1 \pmod{6}$  and  $B = \langle 1 \rangle, \langle 3, 3(\mu - 7)/6 + 4, 6(\mu - 7)/6 + 5 \rangle$  **then**
  - 5:     **if**  $\mu = \tau$  **then**
  - 6:         **return**  $(J_\mu)$ ;
  - 7:     **else**
  - 8:         **return**  $(J_{\mu, \mu - \tau})$ ;
  - 9: **if**  $\mu \equiv 2 \pmod{6}$  and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 2, 2(\mu - 8)/6 + 3 \rangle$  **then**
  - 10:     **if**  $\mu = \tau$  **then**
  - 11:         **return**  $(J_\mu)$ ;
  - 12:     **else**
  - 13:         **return**  $(J_{\mu, \mu - \tau})$ ;
  - 14: **if**  $\mu \equiv 3 \pmod{6}$  and  $B = \langle 1 \rangle, \langle 3, 3(\mu - 9)/6 + 5, 6(\mu - 9)/6 + 7 \rangle$  **then**
  - 15:     **if**  $\mu = \tau$  **then**
  - 16:         **return**  $(J_\mu)$ ;
  - 17:     **else**
  - 18:         **return**  $(J_{\mu, \mu - \tau})$ ;
  - 19: **if**  $\mu \equiv 5 \pmod{6}$  and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$  **then**
  - 20:     **if**  $\mu = \tau$  **then**
  - 21:         **return**  $(J_{(\mu - 5)/6 + 1, 0})$ ;
  - 22: **if**  $\mu \neq \tau$  **then**
  - 23:     **if**  $\mu$  is even and  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 2, 4\tau - 3\mu - 3 \rangle$  **then**
  - 24:         **return**  $(J_{\mu - \tau + 1, 6\tau - 5\mu - 5})$ ;
  - 25:     **if**  $\mu$  is odd **then**
  - 26:         **if**  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle, \langle 1 \rangle$  **then**
  - 27:             **return**  $(J_{\mu - \tau + 1, 6\tau - 5\mu - 5})$ ;
  - 28: **return** *not unimodular*;
-

Using the coordinate change  $x \rightarrow \xi^7 x, y \rightarrow \xi^3 y, z \rightarrow \xi^5 z$ , we may assume  $I_{\lambda_6} = \langle xy + z^2, x^2 + zy^3 + \xi\lambda_6 y^5 \rangle$ . Choose  $\xi$  such that  $\xi\lambda_6 = 1$ . So we obtain  $\langle xy + z^2, x^2 + zy^3 + y^5 \rangle$ . It has 2 branches and same semigroup as  $K_{13}$  but  $\tau = 12$ . It can be differentiated from  $K_{13}$  by the Tjurina number.  $\square$

**Proposition 3.21.** *The unimodular complete intersection curve singularities with Milnor number 14, irreducible having semigroup  $\langle 4, 7, 10 \rangle$  are  $K_{14}$  with Tjurina number 14 defined by the ideal  $\langle xy + z^2, x^2 + y^5 \rangle$  and  $K_{14,1}$  with Tjurina number  $\tau = 13$  defined by the ideal  $\langle xy + z^2, x^2 + y^5 + y^2 z^2 \rangle$ .*

We complete the list of unimodular singularities in this case as

Table 11:

$K_{13,1}$	$\langle xy + z^2, x^2 + zy^3 + y^5 \rangle$	13	12	$\langle 1 \rangle, \langle 3, 5, 7 \rangle$
$K_{14,1}$	$\langle xy + z^2, x^2 + y^5 + y^2 z^2 \rangle$	14	13	$\langle 4, 7, 10 \rangle$

**Proposition 3.22.** *Let  $(V(\langle f, g \rangle), 0) \subseteq (\mathbb{C}^3, 0)$  be the germ of a complete intersection space curve singularity. Assume it is not a hypersurface singularity and the 2-jet of  $\langle f, g \rangle$  has normal form  $\langle xy + z^2, x^2 \rangle$ . Then  $(V(\langle f, g \rangle), 0)$  is unimodular if and only if it is isomorphic to a complete intersection in Table 10 and 11.*

*Proof.* The proof is a direct consequence of C.T.C. Wall's classification and Propositions 3.20 and 3.21.  $\square$

The following Algorithm is the basis for classifying the unimodular complete intersection curve singularities when  $\text{char}(K) = 0$ .

#### 4 Singular examples

```
> ring R=0,(x,y,z),ds;  
> ideal I=xy+11y2+9yz+z3,x2+22xy+121y2+18xz+198yz+81z2+z3+y4;  
> classifyycis1(I);  
HC_13:(xy+z3,x2+z3+y4)  
  
> ideal J=x2+xy+2y2+2xz+z2,x2+2xy+xz+2yz+xy12+y12z;  
> classifyycis1(J);  
J_6*5+8:(xy+z2,xz+zy12)
```

#### References

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**Algorithm 6** Ksingularity(I)

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**Input:**  $I = \langle f, g \rangle \in \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$  having 2-jet  
of the form  $(xy + z^2, x^2)$

**Output:** the type of the singularity

```

1: compute  $\mu$  =Milnor number of the  $I$ ;
2: compute  $\tau$  =Tjurina number of the  $I$ ;
3: compute  $B$  =semigroups of  $I$  corresponding to the branches;
4: if  $\mu = 13$  and  $B = \langle 1 \rangle, \langle 3, 5, 7 \rangle$  then
5:   if  $\mu = \tau$  then
6:     return  $(K_{13})$ ;
7:   if  $\mu - \tau = 1$  then
8:     return  $(K_{13,1})$ ;
9: if  $\mu = 14$  and  $B = \langle 4, 7, 10 \rangle$  then
10:  if  $\mu = \tau$  then
11:    return  $(K_{14})$ ;
12:  if  $\mu - \tau = 1$  then
13:    return  $(K_{14,1})$ ;
14: if  $\mu = 11$  and  $B = \langle 2, 3 \rangle, \langle 2, 3 \rangle$  then
15:  if  $\mu = \tau$  then
16:    return  $(K_{1,0})$ ;
17: if  $\mu \neq \tau$  then
18:  if  $\mu - \tau = 1$  and  $\mu > 11$  then
19:    if  $\mu$  is even then
20:      if  $B = \langle 1 \rangle, \langle 1 \rangle, \langle 2, 3 \rangle$  then
21:        return  $(K_{1,\mu-11})$ ;
22:      if  $B = \langle 2, 3 \rangle, \langle 2, 3 \rangle$  then
23:        return  $(K_{1,\mu-11})$ ;
24:    if  $\mu$  is odd then
25:      if  $B = \langle 2, 3 \rangle, \langle 2, \mu - 8 \rangle$  then
26:        return  $(K_{1,\mu-11})$ ;
27:      if  $B = \langle 4, 6, \mu - 3 \rangle$  then
28:        return  $(K_{1,\mu-11})$ ;
29: return not unimodular;

```

---

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**Algorithm 7** `classifycis1(I)` [Unimodular curve singularities]

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**Input:**  $I = \langle f, g \rangle \subseteq \langle x, y, z \rangle^2 \mathbb{C}[[x, y, z]]$  isolated complete intersection curve singularity.

**Output:** The type of the singularity  $(V(I), 0)$ .

- 1: compute  $I_2$  the 2-jet of  $I$ ;
- 2: compute  $I_2 = \bigcap_{i=1}^s Q_i$  the irredundant primary decomposition over  $\mathbb{C}$ ;
- 3: compute  $d_i = \text{Krull dimension of } \mathbb{C}[x, y, z]/Q_i$ ;
- 4: compute  $h_i \in \mathbb{Q}[t]$  the Hilbert polynomial corresponding to each  $Q_i$ ;
- 5: **if**  $s = 2$  **then**
- 6:     **if**  $d_1 = d_2 = 1$  **then**
- 7:         **if**  $h_1 = h_2 = 2$  **then**
- 8:             **return** `Psingularity(I)`; via Algorithm 1
- 9:         **if**  $h_1 = 1$  and  $h_2 = 4$  **then**
- 10:             **return** `(Jsingularity)`; via Algorithm 5
- 11:     **if**  $d_1 = 1, d_2 = 2$  **then**
- 12:         **if**  $h_1 = 1, h_2 = 1 + t$  **then**
- 13:             **return** `(Fsingularity(I))`; via Algorithm 2
- 14:         **if**  $h_1 = 2$  and  $h_2 = 1 + t$  **then**
- 15:             **return** `(Hsingularity(I))`; via Algorithm 4
- 16: **if**  $s = 1$  **then**
- 17:     compute  $R$  the radical of  $I_2$
- 18:     **if**  $R^3 \not\subseteq I_2$  **then**
- 19:         **return** `(Gsingularity(I))`; via Algorithm 3
- 20:     **else**
- 21:         **return** `(Ksingularity(I))`; via Algorithm 6
- 22: **return** *(not unimodular)*;

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