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Generalized Horadam Polynomials and Numbers

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Abstract

We consider the polynomials $h_{n,m}(x)$ $(m \geq 2)$ and the numbers $h_{n,m}$ $(x = 1)$, which are the generalized Horadam polynomials and the generalized Horadam numbers, respectively. We also consider the polynomials $h^{(s)}_{n,m}(x)$- convolutions of the polynomials $h_{n,m}(x)$, and the sequence of numbers $h^{(s)}_{n,m}$- convolutions of the numbers $h_{n,m}$, where $s \geq 0$.

1 Introduction

In the paper [8] authors considered Horadam polynomials $h_n(x)$, which are given by the following recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad n > 2, \tag{1}$$

with $h_1(x) = a$, $h_2(x) = bx$, $(a, b, p, q$ are some real constants).

We emphasize some particular cases of the polynomials $h_n(x)$:

1° For $a = b = p = q = 1$, we get the Fibonacci polynomials $F_n(x)$;

2° For $a = 2$, $b = p = q = 1$, we get the Lucas polynomials $L_n(x)$;

3° If $a = q = 1$, $b = p = 2$, then we get the Pell polynomials $P_n(x)$;

4° If $a = 1$, $b = p = 2$, $q = -1$, then we get the Chebyshev polynomials of the second kind $U_n(x)$.

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2 Generalized polynomials

In this section we introduce the polynomials $h_{n,m}(x)$ $(m \geq 2)$, the generalized Horadam polynomials, by

$$h_{n,m}(x) = pxh_{n-1,m}(x) + qh_{n-m,m}(x), \quad n > m, \tag{2}$$

with $h_{1,m}(x) = a$, $h_{n,m}(x) = bp^{n-2}x^{n-1}$, for $n = 2, \ldots, m$.

For $x = 1$ in (2), we obtain the generalized Horadam numbers $h_{n,m}$:

$$h_{n,m} = ph_{n-1,m} + qh_{n-m,m}, \quad n > m, \tag{3}$$

with $h_{1,m} = a$, $h_{n,m} = bp^{n-2}$, for $n = 2, \ldots, m$.

Remark 1. For $m = 2$ in (2) and (3), we get Horadam polynomials $h_n(x)$ and Horadam numbers h_n, respectively (see [8]).

Now, using the standard method, starting with the recurrence relation (2), we find that the function

$$F(x,t) = \frac{a + xt(b - ap)}{1 - pxt - qt^m} = \sum_{n=1}^{\infty} h_{n,m}(x)t^{n-1} \tag{4}$$

is the generating function of the polynomials $h_{n,m}(x)$.

Remark 2. For $m = 2$, the relation (4) yields the generating function $g(x,t)$ of the Horadam polynomials $h_n(x)$ (see [8], (13)):

$$g(x,t) = \frac{a + xt(b - ap)}{1 - pxt - qt^2} = \sum_{n=1}^{\infty} h_n(x)t^{n-1}.$$

Using the development of the function $F(x,t)$, given by (4), into the series on t, then comparing the corresponding coefficients to t^n, we obtain the explicit formula for the polynomials $h_{n,m}(x)$

$$\begin{aligned} h_{n,m}(x) =& a \sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k} (px)^{n-1-mk} q^k \\ &+ \left(\frac{b}{p} - a\right) \sum_{k=0}^{[(n-2)/m]} \binom{n-2-(m-1)k}{k} (px)^{n-1-mk} q^k. \end{aligned} \tag{2.4}$$

Remark 3. For $m = 2$ the formula (2.4) yields the explicit formula for Horadam polynomials $h_n(x)$ (see [8], (16)).

Taking $x = 1$ in (2.4), we get the explicit formula for the generalized Horadam numbers $h_{n,m}$:

$$\begin{aligned} h_{n,m} =& ap^{n-1} \sum_{k=0}^{[(n-1)/m]} \binom{n-1-(m-1)k}{k} \left(\frac{q}{p^m}\right)^k \\ &+ p^{n-2}(b-ap) \sum_{k=0}^{[(n-2)/m]} \binom{n-2-(m-1)k}{k} \left(\frac{q}{p^m}\right)^k . \end{aligned} \tag{2.5}$$

Some particular cases of the polynomials $h_{n,m}(x)$ (see [1, 3, 4, 5, 6]) are:

$$F_{n+1,m}(x) = \sum_{k=0}^{[n/m]} \binom{n-(m-1)k}{k} x^{n-mk} \text{ – Fibonacci polynomials ;}$$

$$P_{n+1,m}(x) = \sum_{k=0}^{[n/m]} \binom{n-(m-1)k}{k} (2x)^{n-mk} \text{ – Pell polynomials;}$$

$$J_{n+1,m}(y) = \sum_{k=0}^{[n/m]} \binom{n-(m-1)k}{k} (2y)^{k} \text{ – Jacobsthal polynomials;}$$

$$U_{n,m}(x) = \sum_{k=0}^{[n/m]} (-1)^k \binom{n-(m-1)k}{k} (2x)^{n-mk} \text{ – Chebyshev polynomials.}$$

3 Some properties

Theorem 1. The polynomials $h_{n,m}(x)$ satisfy the following relation

$$\sum_{k=1}^{n-1} h_{k,m}(x) = \frac{h_{n,m}(x) + q\sum_{i=1}^{m-1} h_{n-i,m}(x) - a - x(b-ap)}{px+q-1}. \tag{5}$$

Proof. Starting from the recurrence relation (2) and by the corresponding characteristic equation

$$\lambda^m - px\lambda^{m-1} - q = 0, \tag{6}$$

where λ_i, $i = 1, \dots, m$ are the solutions of the equation (3.2), we have:

$$\begin{aligned}
\sum_{k=1}^{n-1} h_{k,m}(x) &= \sum_{k=1}^{n-1} \left(A_1\lambda_1^{k-1} + A_2\lambda_2^{k-1} + \cdots + A_m\lambda_m^{k-1}\right) \\
&= A_1(1 + \lambda_1 + \lambda_1^2 + \cdots + \lambda_1^{n-2}) \\
&\quad + A_2(1 + \lambda_2 + \lambda_2^2 + \cdots + \lambda_2^{n-2}) + \cdots + \\
&\quad + A_m(1 + \lambda_m + \lambda_m^2 + \cdots + \lambda_m^{n-2}) \\
&= A_1\frac{1-\lambda_1^{n-1}}{1-\lambda_1} + A_2\frac{1-\lambda_2^{n-1}}{1-\lambda_2} + \cdots + A_m\frac{1-\lambda_m^{n-1}}{1-\lambda_m} \\
&= \frac{1}{(1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_m)} \times
\end{aligned}$$

$$\left(A_1(1-\lambda_1^{n-1}\prod_{i=2}^{m}(1-\lambda_i) + A_2(1-\lambda_2^{n-1}\prod_{i=1,i\neq 2}^{m}(1-\lambda_i) + \cdots + A_m(1-\lambda_m^{n-1})\prod_{i=1}^{m-1}(1-\lambda_i)\right).$$

Using the relations (by (3.2))

$$\lambda_1+\lambda_2+\cdots+\lambda_m = px,\ \ \lambda_1\lambda_2+\lambda_1\lambda_3+\cdots+\lambda_{m-1}\lambda_m = 0, \dots (-1)^m\lambda_1\lambda_2\cdots\lambda_m = -q,$$

we get

$$\begin{aligned}
&\left(\sum_{i=1}^{m} A_i(1-\lambda_i^{n-1})\prod_{j=1,j\neq i}^{m}(1-\lambda_j)\right) \cdot ((1-\lambda_1)(1-\lambda_2)\cdots(1-\lambda_m))^{-1} \\
&= \frac{1}{1-px-q} \times \\
&\sum_{i=1}^{m} A_i(1-\lambda_i^{n-1})\left(1-\lambda_i(px-\lambda_i) - \lambda_i^2(px-\lambda_i) + \cdots + (-1)^{m-1}\lambda_i^{m-1}(px-\lambda_i)\right) \\
&= \frac{1}{1-px-q}\left(a - apx + bx - h_{n,m}(x) - q\sum_{i=1}^{m-1} h_{n-i,m}(x)\right).
\end{aligned}$$

□

Remark 4. For $m = 2$ the formula (5) yields (see [8], (18))

$$\sum_{k=1}^{n-1} h_k(x) = \frac{h_n(x) + qh_{n-1}(x) - a - x(b - ap)}{px + q - 1}.$$

Some special cases of the formula (5) are (see [5]):

$$\sum_{k=1}^{n-1} F_{k,m}(x) = \frac{F_{n,m}(x) + \sum_{i=1}^{m-1} F_{n-i,m}(x) - 1}{x},$$

$F_{n,m}(x) -$ the generalized Fibonacci polynomials;

$$\sum_{k=1}^{n-1} P_{k,m}(x) = \frac{P_{n,m}(x) + \sum_{i=1}^{m-1} P_{n-i,m}(x) - 1}{2x},$$

$P_{n,m}(x) -$ the generalized Pell polynomials;

$$\sum_{k=1}^{n-1} U_{k,m}(x) = \frac{U_{n,m}(x) - \sum_{i=1}^{m-1} U_{n-i,m}(x) - 1}{2x - 2},$$

$U_{n,m}(x) -$ the generalized Chebyshev polynomials of the second kind.

4 Convolutions of the generalized Horadam polynomials

In this section we introduce the polynomials $h^{(s)}_{n,m}(x)$, the convolutions of the polynomials $h_{n,m}(x)$, by

$$G(x,t) = \left(\frac{a + xt(b - ap)}{1 - pxt - qt^m}\right)^{s+1} = \sum_{n=1}^{\infty} h^{(s)}_{n,m}(x) t^{n-1}, \tag{7}$$

where $n, m \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$, $n \geq m$.

Starting from (7), we get the following representation of the polynomials $h^{(s)}_{n,m}(x)$:

$$\begin{aligned} h^{(s)}_{n,m}(x) = &\sum_{i=0}^{s+1} \binom{s+1}{i} a^{s+1-i} \left(\frac{b}{p} - a\right)^i \\ &\times \sum_{k=0}^{[(n-1)/m]} \frac{(s+1)_{n-1-i-(m-1)k}}{k!(n-1-i-mk)!} (px)^{n-1-mk} q^k. \end{aligned} \tag{4.2}$$

where $\binom{n}{k} = 0$ for $n < k$.

Some particular cases of the polynomials $h_{n,m}^{(s)}(x)$ are:

1° For $a = b = p = q = 1$, we have $F_{n,m}^{(s)}(x)$, the convolutions of the generalized Fibonacci polynomials and (4.2) becomes

$$F_{n,m}^{(s)}(x) = \sum_{k=0}^{[(n-1)/m]} \frac{(s+1)_{n-1-(m-1)k}}{k!(n-1-mk)!} x^{n-1-mk}. \tag{4.3}$$

If we use the known relations ([7], [4]):

$$(\alpha)_{n+k} = (\alpha)_n(\alpha+n)_k; \quad \frac{(-1)^k}{(n-k)!} = \frac{(-n)_k}{n!}, \ (\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k},$$

then (4.3) takes the following hypergeometric representation

$$F_{n+1,m}^{(s)}(x) = \frac{x^n(s+1)^n}{n!} {}_mF_{m-1}\left[\begin{matrix} \frac{-n}{m}, \frac{1-n}{m}, \dots, \frac{m-1-n}{m}; \ -x^{-m} \\ \frac{-s-n}{m-1}, \frac{1-s-n}{m-1}, \dots, \frac{m-2-s-n}{m-1} \end{matrix}\right].$$

2° For $a = q = 1$, $b = p = 2$, we obtain $P_{n,m}^{(s)}(x)$, the convolutions of the generalized Pell polynomials

$$P_{n+1,m}^{(s)}(x) = \sum_{k=0}^{[n/m]} \frac{(s+1)_{n-(m-1)k}}{k!(n-mk)!} (2x)^{n-mk},$$

or

$$P_{n+1,m}^{(s)}(x) = \frac{(2x)^n(s+1)_n}{n!} {}_mF_{m-1}\left[\begin{matrix} \frac{-n}{m}, \frac{1-n}{m}, \dots, \frac{m-1-n}{m}; \ (2x)^{-m} \\ \frac{-s-n}{m-1}, \frac{1-s-n}{m-1}, \dots, \frac{m-2-s-n}{m-1} \end{matrix}\right].$$

3° For $a = 1$, $b = p = 2$, $q = -1$, in (4.2), we get $U_{n,m}^{(s)}(x)$, the convolutions of the generalized Chebyshev polynomials

$$U_{n,m}^{(s)}(x) = \frac{(2x)^n(s+1)_n}{n!} {}_mF_{m-1}\left[\begin{matrix} \frac{-n}{m}, \frac{1-n}{m}, \dots, \frac{m-1-n}{m}; \ (2x)^{-m} \\ \frac{-s-n}{m-1}, \frac{1-s-n}{m-1}, \dots, \frac{m-2-s-n}{m-1} \end{matrix}\right].$$

Next, for $b = ap$ in (4), we get

$$F(x,t) = a(1-pt-qt^m)^{-1} = \sum_{n=1}^{\infty} h_{n,m}(x)t^{n-1}. \tag{4.3}$$

Differentiating both sides of (4.3) to x, one by one, s-times, we find that

$$h^{(s)}_{n-s,m}(x) = \frac{1}{ap^s s!} D^s h_{n,m}(x), \text{ where } h^s_{n,m}(x) \equiv \frac{\partial^s h_{n,m}(x)}{\partial x^s}. \tag{4.4}$$

Namely, using the formula (4.4), we easily calculate the convolutions of the polynomials $h^{(s)}_{n,m}(x)$. Next, we give the examples for $m = 3$ and $s = 0, 1, 2, 3$, and for $m = 4$ and $s = 0, 1, 2, 3$.

Table 1: $h_{n,3}^{(s)}(x)$

n	$s=0$	$s=1$
1	a	1
2	apx	$2px$
3	$a(px)^2$	$3(px)^2$
4	$a(px)^3+aq$	$4(px)^3+2q$
5	$a(px)^4+2apqx$	$5(px)^4+6pqx$
6	$a(px)^5+3aq(px)^2$	$6(px)^5+12q(px)^2$
7	$a(px)^6+4aq(px)^3+aq^2$	$7(px)^6+20q(px)^3+3q^2$
8	$a(px)^7+5aq(px)^4+3aq^2px$	$8(px)^7+30q(px)^4+12q^2px$
9	$a(px)^8+6aq(px)^5+6aq^2(px)^2$	$9(px)^8+42q(px)^5+30q^2(px)^2$

Table 2: $h_{n,3}^{(s)}(x)$

n	$s=2$	$s=3$
1	1	1
2	$3px$	$4px$
3	$6(px)^2$	$10(px)^2$
4	$10(px)^3+3q$	$20(px)^3+4q$
5	$15(px)^4+12pqx$	$35(px)^4+20pqx$
6	$21(px)^5+30q(px)^2$	$56(px)^5+60q(px)^2$
7	$28(px)^6+60q(px)^3+6q^2$	$84(px)^6+140q(px)^3+10q^2$
8	$36(px)^7+105q(px)^4+30q^2px$	$120(px)^7+280q(px)^4+60q^2px$
9	$45(px)^8+168q(px)^5+90q^2(px)^2$	$165(px)^8+504q(px)^5+210q^2(px)^2$

Table 3: $h_{n,4}^{(s)}(x)$

n	$s=0$	$s=1$
1	a	1
2	apx	$2px$
3	$a(px)^2$	$3(px)^2$
4	$a(px)^3$	$4(px)^3$
5	$a(px)^4+aq$	$5(px)^4+2q$
6	$a(px)^5+2aqpx$	$6(px)^5+6qpx$
7	$a(px)^6+3aq(px)^2$	$7(px)^6+12q(px)^2$
8	$a(px)^7+4aq(px)^3$	$8(px)^7+20q(px)^3$
9	$a(px)^8+5aq(px)^4+aq^2$	$9(px)^8+30q(px)^4+3q^2$

Table 4: $h_{n,4}^{(s)}(x)$

n	$s=2$	$s=3$
1	1	1
2	$3px$	$4px$
3	$6(px)^2$	$10(px)^2$
4	$10(px)^3$	$20(px)^3$
5	$15(px)^4+3q$	$35(px)^4+4q$
6	$21(px)^5+12qpx$	$56(px)^5+20qpx$
7	$28(px)^6+30q(px)^2$	$84(px)^6+60q(px)^2$
8	$36(px)^7+60q(px)^3$	$120(px)^7+140q(px)^3$
9	$45(px)^8+105q(px)^4+6q^2$	$165(px)^8+56q(px)^4+10q^2$

Table 5: $P^{(s)}_{n,4}(x)$-Pell polynomials

n	$s=0$	$s=1$
1	1	1
2	$2x$	$2(2x)$
3	$(2x)^2$	$3(2x)^2$
4	$(2x)^3$	$4(2x)^3$
5	$(2x)^4+1$	$5(2x)^4+2$
6	$(2x)^5+2(2x)$	$6(2x)^5+6(2x)$
7	$(2x)^6+3(2x)^2$	$7(2x)^6+12(2x)^2$
8	$(2x)^7+4(2x)^3$	$8(2x)^7+20(2x)^3$
9	$(2x)^8+5(2x)^4+1$	$9(2x)^8+30(2x)^4+3$

Table 6: $P^{(s)}_{n,4}(x)$

n	$s=2$	$s=3$
1	1	1
2	$3(2x)$	$4(2x)$
3	$6(2x)^2$	$10(2x)^2$
4	$10(2x)^3$	$20(2x)^3$
5	$15(2x)^4+3$	$35(2x)^4+4$
6	$21(2x)^5+12(2x)$	$56(2x)^5+20(2x)$
7	$28(2x)^6+30(2x)^2$	$84(2x)^6+60(2x)^2$
8	$36(2x)^7+60(2x)^3$	$120(2x)^7+140(2x)^3$
9	$45(2x)^8+105(2x)^4+6$	$165(2x)^8+56(2x)^4+10$

Table 7: $P_{n,4}^{(s)}$ -Pell numbers

n	$s=0$	$s=1$	$s=2$	$s=3$
1	1	1	1	1
2	2	4	6	8
3	4	12	24	40
4	8	32	80	160
5	17	82	243	564
6	36	204	696	1832
7	76	7496	1912	5616
8	160	1184	4608	16480
9	337	2787	13206	43146

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