DOI: 10.1515/auom-2017-0046
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 25(3),2017, 215–244

Mixed problem for quasilinear hyperbolic system with coefficients functionally dependent on solution

Małgorzata Zdanowicz and Zbigniew Peradzyński

Abstract

The mixed problem for quasilinear hyperbolic system with coefficients functionally dependent on the solution is studied. We assume that the coefficients are continuous nonlinear operators in the Banach space $C^1(\mathbb{R})$ satisfying some additional assumptions. Under these assumptions we prove the uniqueness and existence of local in time C^1 solution, provided that the initial data are also of class C^1.

1 Introduction

Since the beginning of 70-thies of the last century the Hall effect thrusters are more and more often used in the space technology not only for the correction of the satellites orbits but also as the marching engines in the space missions. Therefore one observes also a violent development of the theoretical studies of Hall thrusters. Although the physics laying behind the construction of such a thruster seems to be simple, there are still important problems and questions which are not yet solved. The rarified neutral gas, usually Xenon, moving through the chamber (a space between two concentric ceramical cylinders) is ionized by collisions with electrons. Neutral gas is released from appropriate

Key Words: Hyperbolic system, differential-functional equations, functional dependence on solution, method of characteristics, mixed problem.

2010 Mathematics Subject Classification: Primary 35L50, 39B, 65M25; Secondary 82D10.

Received: 04.11.2016
Revised: 15.12.2016
Accepted: 21.12.2016

orifices in the anode - the bottom of the chamber, whereas the pimary electrons are produced by the hollow cathode located outside, near the other end (exit) of the chamber. The generated electric field between the anode and the cathode, practically parallel to the axis of chamber is accelerating heavy ions. To have a reasonable thrust, the motion of electrons in the axial directions must be greatly reduced. Otherwise the most of energy from the electric field would be directed to electrons. Therefore a radial magnetic field (perpendicular to the axes of cylinders is applied. As a result, because of the Hall effect, the electrons are subject mainly to the azimuthal motion (Hall current). The motion in the axial direction is of the diffusion type due to collisions of electrons with atoms and the ceramic walls of the channel. One observes also anomalous diffusion caused by the fluctuations of the electric field (plasma turbulence). In the same time the magnetic field is too weak to influence the motion of very heavy ions. In case of Xenon we have $\frac{m_e}{m_i} \approx 10^{-5}$.

The simplest description of plasma discharge in the Hall thruster is based on the 3-fluid model consisting of the fluid of neutral atoms, ions and electrons [1]. The simple geometry allows also to account only for one space variable, assuming cylindrical symmetry and homogeneity of plasma along the radial direction. The distribution of the electric field is dependent on the charge distribution in the chamber and in principle it is governed by the Poisson equation for the electric potential. However typically we have $\frac{n_i - n_e}{n_i} \approx 10^{-5}$. This creates serious difficulties for numerical determination of the particle densities and the electric field. Small errors in the densities leads to large errors in the electric field. This influences the motion and the densities, so the numerical procedure becomes very unstable. The quasineutrality of plasma $n_e \approx n_i$ permits to determine with a good accuracy the electric field by assuming that $n_e = n_i$ and neglecting the Poisson equation. This is common procedure in plasma physics. More precisely assuming in the electron and ion momentum equations $n_e = n_i$ and neglecting the inertial forces in the electron momentum equations one arrives at the Ohm type of equation relating the electric field, electron axial velocity and the gradient of the electron temperature. By the ion and electron continuity equations the plasma neutrality implies that the electric current density $I = en_i(V_i - V_e)$ is independent of the spatial variable x. So it may depend only on time $I = I(t)$. Clearly this can be true as long as the time derivatives of n_e and n_i can be considered as equal.

After inserting so determined electric field to the ion momentum equation one arrives at the following system

- continuity equation for the neutral atoms

$$\frac{\partial N_a}{\partial t} + \frac{\partial (N_a V_a)}{\partial x} = -\beta N_a n_i \tag{1}$$

- continuity equation for ions

$$\frac{\partial n_i}{\partial t} + \frac{\partial (V_i n_i)}{\partial x} = \beta N_a n_i \tag{2}$$

- ion momentum equation

$$\frac{\partial V_i}{\partial t} + V_i \frac{\partial V_i}{\partial x} + \frac{1}{n_i}\frac{\partial}{\partial x}\left(\frac{kT_e}{m_i}\, n_i\right) = \nu_{eff}\left(\frac{I}{n_i\, e} - V_i\right) + \beta N_a (V_a - V_i) \tag{3}$$

- electron temperature equation

$$\frac{n_i}{\sqrt{T_e}}\left[\frac{\partial}{\partial t}\left(\frac{T_e^{\frac{3}{2}}}{n_i}\right) + V_e \frac{\partial}{\partial x}\left(\frac{T_e^{\frac{3}{2}}}{n_i}\right)\right] = Q, \tag{4}$$

$$V_e = V_i - \frac{I}{n_i e}, \tag{5}$$

$$\begin{aligned} Q \quad = \quad & -\frac{\beta N_a}{k}\left(\gamma e E_{ion} + \frac{3}{2} k T_e - E_{ke}\right) \\ & + \frac{2\nu_m}{k} E_{ke} - \frac{\nu_{ew}}{k}\left(E_{ke} + 2kT_e\right) - \beta N_a T_e, \end{aligned} \tag{6}$$

where the total current density I is given by the functional

$$I(t) = \left(\int_0^l \frac{\nu_{eff}}{e\, n_i} dx\right)^{-1} \cdot \left[\frac{e}{m_i} U_0 + \int_0^l \left(\nu_{eff} V_i + \frac{1}{n_i}\frac{\partial}{\partial x}\left(\frac{kT_e}{m_i} n_i\right)\right) dx\right]. \tag{7}$$

The characteristics of the system (1) - (4) have the following slopes:

$$\xi_1 = V_a, \quad \xi_2 = V_i - \sqrt{\frac{5kT_e}{3m_i}}, \quad \xi_3 = V_i + \sqrt{\frac{5kT_e}{3m_i}}, \quad \xi_4 = V_e = V_i - \frac{I}{n_i e}.$$

It is known that $\xi_1 > 0$, $\xi_3 > 0$, $\xi_4 < 0$.
As will be shown Eqs. (1) - (4) form the hyperbolic system. The total current density I is given by (7), therefore both sides of this system depend functionally on the solution.

Besides the initial condition ($x \in [0, l]$):

$$N_a(0,x) = N_{a0}(x),\ n_i(0,x) = n_{i0}(x),\ V_i(0,x) = V_{i0}(x),\ T_e(0,x) = T_{e0}(x),$$

we assign also the boundary conditions. Neutral atoms are moving with the constant velocity V_a. Hence one may think that the ions originated close to the anode should have the same velocity. In such a case the boundary condition on the anode would be in the form $V_i(t,0) = V_a$. Since in reality there is $V_a < \sqrt{\frac{5kT_e}{3m_i}}$, therefore in this case $\xi_2 < 0$ for $x = 0$ and the boundary conditions for system (1) - (4) should be $N_a(t,0) = N_a^*(t)$, $V_i(t,0) = V_a$, $T_e(t,l) = T_e^*(t)$. However, the physical considerations show that in most cases close to the anode, because of the excess of electrons, the anode layer (the sheath) with the reversed electric field is formed. As a result, at the edge of this layer the ions are moving towards the anode with so called Bohm velocity $V_B = \sqrt{\frac{kT_e}{m_i}}$. Consequently, we assume that in this case on the anode ($x = 0$) we have $V_i(t,0) = -\sqrt{\frac{kT_e}{m_i}}$. The number (two) of the boundary conditions on the left boundary is equal to the number of families of characteristics entering the rectangle from the left. Similarly, we only need one boundary condition on the right-hand side, because only ξ_4 is negative there. The thruster is drafted in a such way that outflow is strongly supersonic close to the channel exhaust ($V_i > \sqrt{\frac{5kT_e}{3m_i}}$). Therefore the eigenvalue ξ_2 changes sign in the interior of $[0,T] \times [0,l]$ - it is negative in the vicinity of anode ($\xi_2 < 0$ for $x = 0$) and positive at the end of the channel ($\xi_2 > 0$ for $x = l$). For this reason the second characteristic leaves the left as well as the right boundary and we do not assume any boundary condition related to this characteristic i.e. ξ_2.

We also assume the following consistency conditions that assert continuity of a solution of the considered system:

$$\begin{aligned}
&N_a^*(0) = N_{a0}(0), \\
&N_a^{*\prime}(0) + (N_{a0}V_a)'(0) = -\beta N_{a0}(0) n_{i0}(0),
\end{aligned}$$

$$\begin{aligned}
&T_{e0}(l) = T_e^*(0), \\
&T_e^{*\prime}(0) - \frac{2T_{e0}(l)}{3n_{i0}(l)}\Big(-(V_{i0}n_{i0})'(l) + \beta N_{a0}(l) n_{i0}(l)\Big) + V_e(0,l)T_{e0}'(l) \\
&\qquad\qquad + V_e(0,l)\frac{2T_{e0}(l)}{3n_{i0}(l)} n_{i0}'(l) = \frac{2}{3}Q(0,l),
\end{aligned}$$

$$\begin{aligned}
&V_{i0}(0) = -\sqrt{\frac{kT_{e0}(0)}{m_i}}, \\
&-\sqrt{\frac{k}{4m_iT_{e0}(0)}}T_{e,t}(0,0) + V_{i0}(0)\,V_{i0}'(0) + \frac{k}{m_i}T_{e0}'(0) + \frac{\frac{k}{m}T_{e0}(0)n_{i0}'(0)}{n_{i0}(0)} = \\
&\qquad = \nu_{eff}\left(\frac{I(0)}{n_{i0}(0)\,e} - V_{i0}(0)\right) + \beta N_{a0}(0)(V_a - V_{i0}(0)), \qquad (8)
\end{aligned}$$

where in (8) there is

$$\begin{aligned} T_{e,t}(0,0) &= \frac{2T_{e0}(0)}{3n_{i0}(0)}\Big(- (V_{i0}n_{i0})'(0) + \beta N_{a0}(0)n_{i0}(0)\Big) - V_e(0,0)T'_{e0}(0) \\ &\quad -V_e(0,0)\frac{2T_{e0}(0)}{3n_{i0}(0)}n'_{i0}(0) + \frac{2}{3}Q(0,0). \end{aligned}$$

2 Formulation of the problem

Let X_0, X_1 be the Banach spaces

$$\begin{aligned} X_0 &= \left\{ u \in C([0,l],\mathbb{R}^n); \|u\|_0 := \sup_{x\in[0,l]} \sqrt{\sum_{i=1}^n u_i^2} < \infty \right\} \\ X_1 &= \{u \in C^1([0,l],\mathbb{R}^n); \|u\|_1 := \|u\|_0 + \|u_{,x}\|_0 < \infty\}. \end{aligned}$$

and let $B^1_r(u^0)$ be a closed ball of radius r, centered at u^0 in X_1. We will be concerned with the general quasilinear hyperbolic system of the form

$$u_{,t} + A[u]u_{,x} = b[u], \tag{9}$$

that coefficients are the operators on u. The system (9) is suplemented by the initial condition

$$u(0,x) = u^0(x), \quad x \in [0,l], \tag{10}$$

and boundary conditions defined below.

We confine ourself to the functional dependence with respect to the variable x only. Thus in $A[u]$, $b[u]$ and $D[u]$, $L[u]$, $R[u]$ (that is defined below), u is treated as a function of x, parametrically dependent on t. Similarly we admit that the operators A, b and D, L, R are parametrically dependent on t. We assume also that for a given u from a closed ball $B^1_r(u^0)$, the matrix $A[u]$ ($t \in [0,T]$) has real eigenvalues $\xi_1[u],\ldots,\xi_n[u]$ and can be diagonalized [8], [10]:

$$A[u] = R[u]D[u]L[u], \quad \text{where} \qquad R = L^{-1}$$

$$D[u] = \mathrm{diag}[\xi_1[u],\ldots,\xi_n[u]], \quad L[u] = \begin{bmatrix} L_1[u] \\ \vdots \\ L_n[u] \end{bmatrix}.$$

The rows of the nonsingular matrix $L[u]$ are the left linearly independent eigenvectors of $A[u]$ and the columns of $R[u] = L^{-1}[u]$ are the right eigenvectors of $A[u]$.

For any function u belonging to the closed ball $B^1_r(u^0)$ and $(t,x) \in [0,T] \times [0,l]$ we assume that there are m_1 characteristics entering the rectangle $[0,T] \times [0,l]$ from the left side and $m_2 - m_1$ characteristics entering $[0,T] \times [0,l]$ from the right-hand side:

$$\begin{array}{rcll} \xi_i[u](t,0) & > & 0, & i = 1, \ldots, m_1 \\ \xi_i[u](t,l) & < & 0, & i = m_1 + 1 \ldots, m_2, \qquad m_2 \leq n. \end{array}$$

The rest of the eigenvalues of the matrix $A[u]$, i.e. $\xi_i[u]$ for $i = m_2 + 1, \ldots, n$, satisfy the conditions

$$\xi_i[u](t,0) < 0 \quad \text{for} \quad t \in [0,T] \quad \text{or} \quad \xi_i[u](t,0) = 0 \quad \text{for} \quad t \in [0,T],$$

$$\xi_i[u](t,l) > 0 \quad \text{for} \quad t \in [0,T] \quad \text{or} \quad \xi_i[u](t,l) = 0 \quad \text{for} \quad t \in [0,T].$$

It means that the characteristic belonging to the i-th family, for $i = m_2 + 1, \ldots, n$ do not enter the rectangle through the latteral boundaries of $[0,T] \times [0,l]$ but they can leave it.
We assume that the lines $x = 0$ and $x = l$ are not the characteristics.
Consequently we assume m_1 conditions on the boundary $x = 0$:

$$F_j(t, u(t,0)) = 0, \quad j = 1, \ldots, m_1, \tag{11}$$

and $m_2 - m_1$ conditions if $x = l$:

$$F_j(t, u(t,l)) = 0, \quad j = m_1 + 1, \ldots, m_2. \tag{12}$$

It is required that $F_j \in C^1(\mathbb{R}^{n+1})$, $j = 1, \ldots, m_2$ and are bounded together with their derivatives.
Multiplying (9) on the left by $L[u]$, we obtain the characteristic form of equations

$$L[u]\, u_t + D[u]\, L[u]\, u_x = L[u]\, b[u]. \tag{13}$$

2.1 Initial-boundary value problem

Let $u^0(x)$ be an initial condition (10) for the system (9). We will assume that there exist a closed ball $B^1_r(u^0)$ in X_1 such that for all $t \in [0,T]$ the following conditions hold:

(A_1) $K : B^1_r(u^0) \to X_1$ and for some constant $c < \infty$: $\|K[v]\|_1 \leq c$ for all $v \in B^1_r(u^0)$, where K denotes L, R, D, b.

(A_2) L is a continuous nonlinear operator, $L : B^1_r(u^0) \to X_1$. In addition we assume that L is Fréchet differentiable and $\exists_{c>0} \forall_{v \in B^1_r(u^0)} \|L'[v]\|_0 \leq c$.

(A_3) $L[v]$ is of C^1 class with respect to the parameter t and there is a constant c such that $\|\frac{\partial}{\partial t}L[v]\|_0 \leq c$, $v \in B^1_r(u^0)$. *

(A_4) For any $\delta > 0$ and $|x - \bar{x}| \leq \delta$ there is a constant c and a function $N(\delta)$, $N(\delta) \to 0$ as $\delta \to 0$ such that for all $v \in B^1_r(u^0)$ there is

$$\left|\frac{\partial}{\partial x}K[v](t,x) - \frac{\partial}{\partial x}K[v](t,\bar{x})\right| \leq c|v_{,x}(x) - v_{,x}(\bar{x})| + N(\delta),$$

for fixed $t \in [0,T]$. Here K stands for L, D, b. $|\cdot|$ denotes the Euclidean metric.

(A_5) There exists a constant c that $\|K[v] - K[\bar{v}]\|_0 \leq c\|v - \bar{v}\|_0$ for $v, \bar{v} \in B^1_r(u^0)$, where K stands for L, R, D, b.

We also assume the following consistence conditions (that assert continuity of solution and its derivatives):

- the consistence condition for the initial condition at $x = 0$ and $x = l$

$$\begin{aligned} F_j(0, u^0(0)) &= 0, \quad j = 1, \ldots, m_1, \\ F_j(0, u^0(l)) &= 0, \quad j = m_1 + 1, \ldots, m_2, \end{aligned}$$

- the consistence condition for the derivatives
for $j = 1, \ldots, m_1$

$$F_{j,t}(0, u^0(0)) + \sum_{i=1}^{n} F_{j,u_i}(0, u^0(0)) \cdot \Big(b[u^0](0) - A[u^0](0)\ u^0_{,x}(0)\Big)_i = 0,$$

for $j = m_1 + 1, \ldots, m_2$

$$F_{j,t}(0, u^0(l)) + \sum_{i=1}^{n} F_{j,u_i}(0, u^0(l)) \cdot \Big(b[u^0](l) - A[u^0](l)\ u^0_{,x}(l)\Big)_i = 0.$$

Let we denote the column vectors $\tilde{F} := [F_i]_{i=1,\ldots,m_1}$, $\tilde{\tilde{F}} := [F_i]_{i=m_1+1,\ldots,m_2}$, and the matrices consisted of the elements of R: $[R_{ij}]_{\substack{i=1,\ldots,n \\ j=1\ldots,m_1}}$, $[R_{ij}]_{\substack{i=1,\ldots,n \\ j=m_1+1\ldots,m_2}}$. Then for $x = 0$ we require:

$$\det\Big(\tilde{F}_{,u}[R_{ij}]_{\substack{i=1,\ldots,n \\ j=1\ldots,m_1}}\Big) \neq 0, \tag{14}$$

*The derivative in (A_3) is the partial derivative with respect to t for the mapping $(t,v) \to L[v]$, where $(t,v) \in [0,T] \times B^1_r(u^0)$ (function v is independent of t). If $u = u(t,x)$, then we write the partial derivative of the operator L with respect to t in u as $\left(\frac{\partial}{\partial t}L\right)[u]$, hence $\left(\frac{\partial}{\partial t}L\right)[u] = \frac{\partial}{\partial t}L[v]\Big|_{v=u}$. In the other hand $\frac{\partial}{\partial t}(L[u])$ is a sum of the partial derivative $\left(\frac{\partial}{\partial t}L\right)[u]$ and the Fréchet derivative $L'[u]$ acting on $u_{,t}$: $\frac{\partial}{\partial t}(L[u]) = \left(\frac{\partial}{\partial t}L\right)[u] + L'[u]u_{,t}$.

and for $x = l$

$$\det\left(\tilde{\tilde{F}}_{,u}[R_{ij}]_{\substack{i=1,\dots,n \\ j=m_1+1\dots,m_2}}\right) \neq 0, \tag{15}$$

where $\tilde{F}_{,u}$ means the matrix $\left[\frac{\partial F_i}{\partial u_j}\right]_{\substack{i=1,\dots,m_1 \\ j=1,\dots,n}}$ and analogously $\tilde{\tilde{F}}_{,u}$ is the matrix $\left[\frac{\partial F_i}{\partial u_j}\right]_{\substack{i=m_1+1,\dots,m_2 \\ j=1,\dots,n}}$.
When we consider the prolonged system, the above two conditions will enable us to calculate appropriate invariants on the boundary.

Theorem 1. *Under the conditions stated above, there exits a local in time, unique solution of class C^1 of the problem (9) - (10), (11) - (12).*

The proof of the theorem basis on the reasoning contained in [11]. It is worth pointing out that many results for quasilinear hyperbolic systems have been studied in [2], [3], [4] [5], [6], [7].

3 Characteristics

The proof of Theorem 1 we begin with the definition of characteristic.

Definition 1. *The characteristic curve $x = x_k(t; \bar{t}, \bar{x})$ of the k-th family coming to the point $(\bar{t}, \bar{x})$ is the solution of the problem*

$$\frac{dx}{dt} = \xi_k[u](t, x), \quad t \in [0, \bar{t}], \tag{16}$$

$$x_k(t; \bar{t}, \bar{x})|_{t=\bar{t}} = \bar{x}. \tag{17}$$

For $u \in B^1_r(u^0)$ the function $\xi_k[u](t, x)$ is bounded and has bounded derivative with respect to x. Hence it satisfies the Lipschitz condition with respect to x and therefore initial problem (16)-(17) has a unique solution (by Picard's theorem). Through each point $(\bar{t}, \bar{x}) \in [0, T^*] \times [0, l]$ ($T^* \leq T$ is given below by (32) on the page 227) there passes one and only one characteristic of the k-th family, which is defined for $t \in [0, T^*]$.

Now we define the characteristics starting from the points $(0, 0)$ and $(l, 0)$. Let $x = \Phi_i(t)$, $i = 1, \dots, m_1$ be a solution of the problem $\frac{dx}{dt} = \xi_i[u](t, x)$, $\Phi_i(0) = 0$, $i = 1, \dots, m_1$, and $x = \Phi_i(t)$, $i = m_1 + 1, \dots, m_2$, be a solution of the problem $\frac{dx}{dt} = \xi_i[u](t, x)$, $\Phi_i(0) = l$, $i = m_1 + 1, \dots, m_2$. Characteristic $x = x_i(t; \bar{t}, \bar{x})$, $i = 1, \dots, m_2$ is continuously differentiable function with respect to t and moreover $\frac{dx_i}{dt} = \xi_i[u](t, x) > 0$ for $i = 1, \dots, m_1$ and $\frac{dx_i}{dt} = \xi_i[u](t, x) < 0$ for $i = m_1+1, \dots, m_2$ in the rectangle $[0, T] \times [0, l]$. From

the equation $0 = x_i(t;\bar{t},\bar{x})$, $i = 1,\dots,m_2$ we can (using implicit function theorem) uniquely calculate t as $t = \sigma_i(\bar{t},\bar{x})$. For $i = 1,\dots,m_1$ it is the time for which the characteristic passing through the point $(\bar{t},\bar{x})$, where $\bar{x} < \Phi_i(\bar{t})$, crosses the boundary $x = 0$. For $i = m_1+1,\dots,m_2$ it is the time for which the characteristic passing through the point $(\bar{t},\bar{x})$, where $\bar{x} > \Phi_i(\bar{t})$, crosses the boundary $x = l$.
Function σ_i, $i = 1,\dots,m_2$ is continuous in the rectangle $[0,T]\times[0,l]$ and $x_i(\sigma_i(\bar{t},\bar{x});\bar{t},\bar{x}) = 0$. Besides $\sigma_i(\bar{t},0) = \bar{t}$ for $i = 1,\dots,m_1$ and $\sigma_i(\bar{t},l) = \bar{t}$ for $i = m_1+1,\dots,m_2$. Let us noticed that the characteristic $x = \Phi_i(t)$, $i = 1,\dots,m_2$ divides the rectangle $[0,T]\times[0,l]$ into two parts and in each part the solution is determined in a different way. Define sets

- $i = 1,\dots,m_1$

$$\begin{aligned} G_{piT} &= \{(t,x)\in[0,T]\times[0,l]:\ x \geq \Phi_i(t)\}, \\ G_{biT} &= \{(t,x)\in[0,T]\times[0,l]:\ x \leq \Phi_i(t)\}, \end{aligned}$$

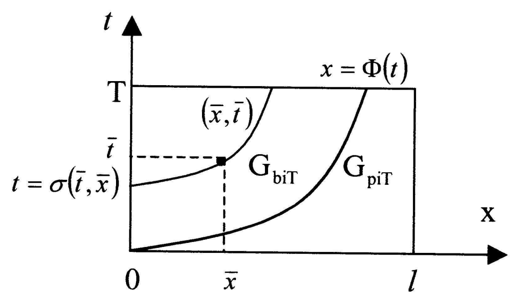

Figure 1: Example of G_{piT} and G_{biT} for $i = 1,\dots,m_1$.

- $i = m_1+1,\dots,m_2$

$$\begin{aligned} G_{piT} &= \{(t,x)\in[0,T]\times[0,l]:\ x \leq \Phi_i(t)\}, \\ G_{biT} &= \{(t,x)\in[0,T]\times[0,l]:\ x \geq \Phi_i(t)\}, \end{aligned}$$

- $i = m_2+1,\dots,n$

$$\begin{aligned} G_{piT} &= [0,T]\times[0,l], \\ G_{biT} &= \varnothing. \end{aligned}$$

4 Prolonged system

Let us define the prolongation of system (9) which will help us to estimate the growth of solution of system (13) as well as its derivatives.
We introduce the new unknown vector function p by

Definition 2.

$$p(t,x) = L[u(t,\cdot)]\ u_{,x}. \tag{18}$$

We will use the following denotation for $v \in X_1$ independent of t:

$$L_{,t}[v] = \frac{\partial}{\partial t} L[v], \tag{19}$$

Thus if $u = u(t,x)$ then

$$L_{,t}[u] = \left. \frac{\partial}{\partial t} L[v] \right|_{v=u}. \tag{20}$$

The Frechét derivative of $L[u]$ acting on ω will be denoted by

$$L'(u;\omega) := L'[u]\omega, \quad u \in X_1,\ \omega \in X_0.$$

Now we formally differentiate all equations (13) with respect to x and we obtain

$$\left(\frac{\partial}{\partial x} L[u]\right) u_{,t} + L[u]u_{,tx} + \left(\frac{\partial}{\partial x} D[u]\right) L[u]u_{,x} + D[u]\frac{\partial}{\partial x}(L[u]u_{,x}) = \frac{\partial}{\partial x}\left(L[u]\, b[u]\right).$$

For the derivative $u_{,tx}$ we have $L[u]\, u_{,tx} = \frac{\partial}{\partial t}\left(L[u]\, u_{,x}\right) - \frac{\partial}{\partial t}\left(L[u]\right) u_{,x}$ where by assumption (A_2) and by (20) we can develop $\frac{\partial}{\partial t}(L[u])$ as $\frac{\partial}{\partial t}(L[u]) = L'(u; u_{,t}) + L_{,t}[u]$. Finally, expressing $u_{,t}$ and $u_{,x}$ from (9) and (18) in terms of p we obtain the prolonged system:

$$u_t = b[u] - R[u]D[u]p, \tag{21}$$

$$\begin{aligned}\frac{\partial p}{\partial t} + D[u]\,\frac{\partial p}{\partial x} = L[u]\frac{\partial}{\partial x} b[u] + \Bigg[\left(\frac{\partial}{\partial x} L[u]\right) R[u]D[u] \\ + L'\Big(u; b[u] - R[u]D[u]p\Big) R[u] + L_{,t}[u]R[u] - \frac{\partial}{\partial x} D[u]\Bigg]\, p,\end{aligned} \tag{22}$$

$$u(0,x) = u^0(x), \tag{23}$$

$$p(0,x) = p^0(x) = L[u^0]\,\frac{du^0}{dx}. \tag{24}$$

Now we will consider the boundary conditions for the prolonged system. Differentiating the boundary condition (11) with respect to t and expressing $u_{i,t}$

from (21), we get (for $j = 1, \ldots, m_1$, at the point $(t, 0)$) $\tilde{F}_{,u} RDp = \tilde{F}_{,t} + \tilde{F}_{,u} b$. By assumption (14) and implicit function theorem, we are able to calculate invariants $p_1(t, 0), \ldots, p_{m_1}(t, 0)$:

$$\begin{bmatrix} p_1(t,0) \\ \vdots \\ p_{m_1}(t,0) \end{bmatrix} = \left\{ \tilde{F}_{,u} \left[R_{ij}\right]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m_1}} \left[D_{ij}\right]_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,m_1}} \right\}^{-1}_{(t,0)} \times \tag{25}$$
$$\times \left\{ \tilde{F}_{,t} + \tilde{F}_{,u} b - \tilde{F}_{,u} \left[R_{ij}\right]_{\substack{i=1,\ldots,n \\ j=m_1+1,\ldots,n}} \left[D_{ij}\right]_{\substack{i=m_1+1,\ldots,n \\ j=m_1+1,\ldots,n}} \begin{bmatrix} p_{m_1+1} \\ \vdots \\ p_n \end{bmatrix} \right\}_{(t,0)}$$

and similarly for $p_{m_1+1}(t, l), \ldots, p_{m_2}(t, l)$.
It is worth pointing out that system (21)-(22) is expressed in Riemann invariants, i.e. it has a diagonal form, whereas (9), in general, is not.

From now on for any matrix $[M_{ij}(t, x)]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$ we will denote by M_k the k-tk row of the matrix.
In order to simplify the notation we define the following operator

Definition 3 (substitution operator $\mathcal{P}$)**.** *For a matrix function* $f(t, x) = [f_{ij}(t, x)]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$ *we define the linear mapping* $\mathcal{P}$*:*

$$\mathcal{P} : \Big(C([0, T] \times [0, l])\Big)^{nm} \longrightarrow \Big(C([0, T] \times [0, T] \times [0, l])\Big)^{nm},$$

$$(\mathcal{P}f)_k(t, \bar{t}, \bar{x}) = [f_{k1}(t, x_k(t; \bar{t}, \bar{x})), \ldots, f_{km}(t, x_k(t; \bar{t}, \bar{x}))], \quad k = 1, \ldots, n.$$

Thus $\mathcal{P}$ acts in this way that in the k-th row ($k = 1, \ldots, n$) of the matrix function $f(t, x)$ it substitutes for x the expression of the k-th family of characteristics $x_k(t; \bar{t}, \bar{x})$.
Since $\sup\limits_{(t,\bar{t},\bar{x}) \in [0,T]\times[0,T]\times[0,l]} |f_k(t, x_k(t; \bar{t}, \bar{x}))| = \sup\limits_{(t,x)\in[0,T]\times[0,l]} |f_k(t, x)|$, then $\mathcal{P}$ is bounded and hence continuous. For the convenience we will use the following denotation: $\mathcal{P}_t f = (\mathcal{P}f)(t, \cdot, \cdot)$.

Let us notice that the left-hand side of (22) is the directional derivative along the characteristic curves:

$$\frac{d}{d\,t}(\mathcal{P}_t p) = \mathcal{P}_t f, \tag{26}$$

where $f = L[u] \frac{\partial}{\partial x} b[u] + \left[\left(\frac{\partial}{\partial x} L[u] \right) R[u] D[u] + L' \Big(u; b[u] - R[u] D[u] p \Big) R[u] + L_{,t}[u] R[u] - \frac{\partial}{\partial x} D[u] \right] p$. Integrating (26) along characteristics with respect to t we obtain:

- for $(\bar{t},\bar{x}) \in G_{piT}$, $i = 1,\ldots,n$, including the initial condition

$$p_i(\bar{t},\bar{x}) = (\mathcal{P}_0 p)_i + \int_0^{\bar{t}} (\mathcal{P}_t f)_i \, dt. \tag{27}$$

- for $(\bar{t},\bar{x}) \in G_{biT}$, $i = 1,\ldots,m_1$, including the boundary condition for $x = 0$

$$p_i(\bar{t},\bar{x}) = p_i(\sigma_i, 0) + \int_{\sigma_i}^{\bar{t}} (\mathcal{P}_t f)_i \, dt. \tag{28}$$

- for $(\bar{t},\bar{x}) \in G_{biT}$, $i = m_1+1,\ldots,m_2$, including the boundary condition for $x = l$

$$p_i(\bar{t},\bar{x}) = p_i(\sigma_i, l) + \int_{\sigma_i}^{\bar{t}} (\mathcal{P}_t f)_i \, dt. \tag{29}$$

To derive Eqs. (21)-(22) we need, in principle, to assume that $u(t,x) \in C^2$. However, the integral form (27), (28), (29) permits us to look for solutions which are only continuous, although p_k is differentiable along the k-th characteristics ($k = 1,\ldots,n$).
Let (u,p) belong to the space $X_0 \times X_0$ with norm $\|(u,p)\|_* := (c^3+1)\|u\|_0 + c\|p\|_0$. If (u,p) is in a ball $B^*_\rho(u^0,p^0)$ centered at (u^0,p^0) and open in $X_0 \times X_0$, then function u stays in the ball $B^1_r(u^0)$ for enough small ρ. From assumptions (A_2) and (A_5) we get $\|u_{,x} - u^0_{,x}\|_0 = \|R[u]\,p - R[u^0]\,p^0\|_0 \le c\|p-p^0\|_0 + c\|u - u^0\|_0\|p^0\|_0$. Since $\|p^0\|_0 = \left\|L[u^0]\frac{du^0}{dx}\right\|_0 \le c^2$, we have $\|u_{,x} - u^0_{,x}\|_0 + \|u-u^0\|_0 \le (c^3+1)\|u-u^0\|_0 + c\|p-p^0\|_0 < r$.

Now we will show that if there exists a solution $(u(t,x),p(t,x))$ of Eqs. (21)-(22) (p in the sense of Eq. (27), (28) or (29)) then it must stay in $B^*_\rho(u^0,p^0)$ for some finite time $t \in [0,T^*]$, where T^* is defined by (32).
The following estimations hold in the ball $B^*_\rho(u^0,p^0)$:

$$|u_{,t}(t,x)| \le \|b[u]\|_0 + \|R[u]\|_0 \, \|D[u]\|_0 \, \|p\|_0 \le c + c^2\|p\|_0.$$

Using $\|p\|_0 \le \|p-p^0\|_0 + \|p^0\|_0 \le \frac{r}{c} + c^2$, we get

$$|u_{,t}(t,x)| \le c_u, \tag{30}$$

where $c_u := c + c^4 + cr$.
Function p_k is differentiable along the k-th characteristics. Then by (22) we can write $\left|\frac{d}{dt}(\mathcal{P}_t p)\right| \le c^2 + \|p\|_0(c+c^2+2c^3) + c^4\|p\|_0^2$. Hence

$$\left|\frac{d}{dt}(\mathcal{P}_t p)\right| \le c_p, \tag{31}$$

where $c_p := c^2 + c^3 + c^4 + 2c^5 + c^8 + r(1 + c + 2c^2 + 2c^5) + r^2c^2$.
As for any function $\varphi(t) \in C^1$ there is $\frac{d}{dt}|\varphi(t)| \leq \left|\frac{d}{dt}\varphi(t)\right|$. Then we obtain from (30), (31) conditions:

$$\frac{\partial}{\partial t}|u(t,x) - u^0(x)| \leq c_u, \quad \frac{\partial}{\partial t}|\mathcal{P}_t p - \mathcal{P}_0 p| \leq c_p,$$

which imply

$$|u(t,x) - u^0(x)| \leq t\, c_u, \quad |\mathcal{P}_t p - \mathcal{P}_0 p| \leq t\, c_p.$$

Because c_u, c_p are constants independent of x therefore we have

$$\|(u,p) - (u^0,p^0)\|_* = (c^3+1)\|u - u^0\|_0 + c\|p - p^0\|_0 \leq t\left(c_u(c^3+1) + c_p\, c\right).$$

If

$$T^* = \min\left\{\frac{r}{c_u(c^3+1) + c_p\, c}, T\right\}, \tag{32}$$

then we see that solution must indeed stay in $B^*_\rho(u^0,p^0)$ (hence it is bounded) for $t \in [0, T^*]$.

5 Uniqueness of solution

We shall show the following

Lemma 1. *If there exists a solution of the mixed problem (9) - (10), (11) - (12), then it is unique.*

Proof. Assume $u(t,x)$ and $\bar{u}(t,x)$ are two different solutions of problem (9) - (10), (11) - (12), and moreover $u(0,x) = \bar{u}(0,x) = u^0(x)$. For abbreviation we will write $\bar{L} = L[\bar{u}]$, $\bar{D} = D[\bar{u}]$, $\bar{b} = b[\bar{u}]$, $\bar{R} = R[\bar{u}]$. We form the difference $v(t,x) = u(t,x) - \bar{u}(t,x)$, $v(0,x) = [0,\ldots,0]^T$, for which holds

$$\bar{L}v_{,t} + \bar{D}\bar{L}v_{,x} = Lb - \bar{L}\bar{b} - (L - \bar{L})u_{,t} - (DL - \bar{D}\bar{L})u_{,x}. \tag{33}$$

The form (33) of the system suggests introducing a new unknown function $\bar{v} = \bar{L}\, v$, and then we may write the system (33) in the Riemann invariants

$$\frac{\partial \bar{v}}{\partial t} + \bar{D}\,\frac{\partial \bar{v}}{\partial x} = g \tag{34}$$

where

$$\begin{aligned} g \;\; &= \;\; Lb - \bar{L}\bar{b} - (L - \bar{L})\, u_{,t} - (D\, L - \bar{D}\, \bar{L})\, u_{,x} \\ &\quad + \left(L_{,t}[\bar{u}] + L'(\bar{u}; \bar{b} - \bar{R}\,\bar{D}\,\bar{p}) + \bar{D}\,\frac{\partial}{\partial x}\bar{L}\right)\bar{R}\,\bar{v}. \end{aligned}$$

Hence we have $\frac{d}{dt}(\mathcal{P}_t \bar{v}) = \mathcal{P}_t g$. After integrating from 0 to $\bar{t}$ we obtain the following integral system:

- if $(\bar{t},\bar{x}) \in G_{piT_0}$, $i = 1,\ldots,n$, then we take into account the initial condition $\bar{v}_i(0,x) = 0$

$$\bar{v}_i(\bar{t},\bar{x}) = \int_0^{\bar{t}} (\mathcal{P}_t g)_i \, dt,$$

- if $(\bar{t},\bar{x}) \in G_{biT}$, $i = 1,\ldots,m_1$ then we take into account the boundary condition for $x = 0$:

$$\bar{v}_i(\bar{t},\bar{x}) = \bar{v}_i(\sigma_i,0) + \int_{\sigma_i}^{\bar{t}} (\mathcal{P}_t g)_i \, dt,$$

- if $(\bar{t},\bar{x}) \in G_{biT}$, $i = m_1+1,\ldots,m_2$, then we consider the boundary condition for $x = l$:

$$\bar{v}_i(\bar{t},\bar{x}) = \bar{v}_i(\sigma_i,l) + \int_{\sigma_i}^{\bar{t}} (\mathcal{P}_t g)_i \, dt,$$

For $(\bar{t},\bar{x}) \in G_{piT}$, $i = 1,\ldots,n$ (i.e. at the point belonging to the set where the initial problem is considered) we obtain

$$\begin{aligned}
&|\bar{v}_i(\bar{t},\bar{x})| \le \int_0^{\bar{t}} \left\|Lb - \bar{L}\bar{b}\right\|_0 \, dt + \int_0^{\bar{t}} \left\|L - \bar{L}\right\|_0 \|u_t\|_0 \, dt \\
&+ \int_0^{\bar{t}} \left\|D\,L - \bar{D}\,\bar{L}\right\|_0 \|u_x\|_0 \, dt + \int_0^{\bar{t}} \|\bar{L}_{,t}\|_0 \left\|\bar{R}\right\|_0 \|\bar{v}\|_0 \, dt \\
&+ \int_0^{\bar{t}} \left\|L'\left(\bar{u};\bar{b} - \bar{R}\,\bar{D}\,\bar{p}\right)\right\|_0 \left\|\bar{R}\right\|_0 \|\bar{v}\|_0 \, dt + \int_0^{\bar{t}} \left\|\bar{D}\right\|_0 \left\|\frac{\partial}{\partial x}\bar{L}\right\|_0 \left\|\bar{R}\right\|_0 \|\bar{v}\|_0 \, dt.
\end{aligned}$$

Now we easily arrive at the following estimations:

- $\|u_{,x}(t,x)\|_0 = \|R[u]\,p\|_0 \le c\,\|p\|_0 \le r + c^3$,
- $\|L'(\bar{u};\bar{u}_t)\|_0 = \|L'\left(\bar{u};b[u] - R[u]\,D[u]\,p\right)\|_0 \le c\,\|b[u] - R[u]\,D[u]\,p\|_0 \le c^2 + c^5 + c^2 r$,

As a consequence of these inequalities and assumptions $(A_1),(A_3),(A_5)$ we can write $(i = 1,\ldots,n)$

$$\begin{aligned}
&|\bar{v}_i(\bar{t},\bar{x})| \le c_1 \int_0^{\bar{t}} \|u(t,x) - \bar{u}(t,x)\|_0 \, dt + c_2 \int_0^{\bar{t}} \|\bar{v}(t,x)\|_0 \, dt \qquad (35) \\
&\le c_1 \int_0^{\bar{t}} \|\bar{R}[u](t,x)\|_0 \|\bar{v}(t,x)\|_0 \, dt + c_2 \int_0^{\bar{t}} \|\bar{v}(t,x)\|_0 \, dt \le c_3 \int_0^{\bar{t}} \|\bar{v}(t,x)\|_0 \, dt,
\end{aligned}$$

where $c_1 = 3c^2(1 + c^2 + r)$, $c_2 = c^2 + 2c^3 + c^6 + c^3 r$, $c_3 = c_1 c + c_2$.
We next claim that the following inequalities hold

$$\begin{aligned} |\bar{v}_i(\sigma_i, 0)| &\leq \tilde{c}(\bar{t}) \sup_{t\in[0,\bar{t}]} \|\bar{v}(t,x)\|_0, \qquad i = 1,\dots,m_1 && (36)\\ |\bar{v}_i(\sigma_i, l)| &\leq \tilde{c}(\bar{t}) \sup_{t\in[0,\bar{t}]} \|\bar{v}(t,x)\|_0, \qquad i = m_1+1,\dots,m_2 && (37)\end{aligned}$$

for some nonnegative constant $\tilde{c}$ dependent on $\bar{t}$ (in σ_i) and $\tilde{c} \to 0$, if $\bar{t} \to 0$.
Proof of (36):
The difference $F_j(t, u(t,0)) - F_j(t, \bar{u}(t,0)) = 0$, for $j = 1,\dots,m_1$ we can rewrite by Hadamard's lemma in the form

$$0 = \sum_{k=1}^{n} \Big(u_k(t,0) - \bar{u}_k(t,0)\Big) \int_0^1 \frac{\partial F_j}{\partial u_k}\Big(t, \bar{u}_1(t,0) + \lambda v_1(t,0), \dots, \bar{u}_n(t,0) + \lambda v_n(t,0)\Big)\, d\lambda. \tag{38}$$

Set the matrix $\Psi = [\psi_{jk}]_{\substack{j=1,\dots,m_1\\ k=1,\dots,n}}$, where

$$\psi_{jk}(t) = \int_0^1 \frac{\partial F_j}{\partial u_k}\Big(t, \bar{u}_1(t,0) + \lambda v_1(t,0), \dots, \bar{u}_n(t,0) + \lambda v_n(t,0)\Big)\, d\lambda.$$

Then (38) is in the form $0 = \Psi(t)\, v(t,0)$. Now we can rewrite the above equality using function $\bar{v}$: $0 = \Psi(t)\bar{R}(t,0)\bar{v}(t,0)$. In order to determine $\bar{v}_1(t,0),\dots,\bar{v}_{m_1}(t,0)$ it must be satisfied the condition $\det\Big(\Psi\, [\bar{R}_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m_1}}\Big) \neq 0$, for $x = 0$. This gives

$$\begin{bmatrix} \bar{v}_1(t,0) \\ \vdots \\ \bar{v}_{m_1}(t,0) \end{bmatrix} = -\Big(\Psi\, [\bar{R}_{ij}]_{\substack{i=1\dots n\\ j=1\dots m_1}}\Big)^{-1} \Psi\, [\bar{R}_{ij}]_{\substack{i=1\dots n\\ j=m_1+1\dots n}} \begin{bmatrix} \bar{v}_{m_1+1}(t,0) \\ \vdots \\ \bar{v}_n(t,0) \end{bmatrix}. \tag{39}$$

For $i = 1,\dots,m_1$ there is $(\sigma_i, 0) \in G_{pjT_0}$ for $j = m_1+1,\dots,n$. Thus we conclude from (35) that $|\bar{v}_j(\sigma_i,0)| \leq \sigma_i\, c_3 \sup_{t\in[0,\bar{t}]} \|\bar{v}(t,x)\|_0$. On account of (39) and the boundedness of the functions Ψ, $\bar{R}$ and the above inequality we have (36). The constant $\tilde{c}(\bar{t})$ is expressed by σ_i, $i = 1,\dots,m_1$, and hence also by $\bar{t}$, whereas $\sigma_i(\bar{t},\bar{x}) \to 0$ if $\bar{t} \to 0$.

□

According to (35), (36), (37) we have for any $(\bar{t},\bar{x}) \in [0,T] \times [0,l]$

$$\|\bar{v}(\bar{t},\bar{x})\|_0 \leq \tilde{c}(\bar{t}) \sup_{t\in[0,\bar{t}]} \|\bar{v}(t,x)\|_0 + c_3 \int_0^{\bar{t}} \|\bar{v}(t,x)\|_0 dt. \tag{40}$$

If $\bar{t}$ satisfies the condition

$$\tilde{c}(\bar{t}) < 1, \tag{41}$$

then we can rewrite (40) as $\|\bar{v}(\bar{t},\bar{x})\|_0 \leq \frac{c_3}{1-\tilde{c}} \int_0^{\bar{t}} \|\bar{v}(t,x)\|_0 dt$. The Gronwall lemma now yields $\|\bar{v}(\bar{t},\bar{x})\|_0 = 0$. Hence the solution is unique if $\bar{t}$ satisfies (41). Then taking $\bar{t}$ as the initial time we easily draw the same conclusion for the next segment of time. We follow by the same method as long as we reach the maximal time of the existence of the solution. It is a correct reasoning, because all constants from the page 228 do not change in the domain of determinacy of the problem (9) - (10), (11) - (12).

6 Existence of solution

To prove the existence of the solution of (9) - (10), (11) - (12) we use the method of successive approximations.
For abbreviation we will write $L = L[u]$ and $\bar{L} = L[\bar{u}]$ and similarly for the other operators.
We define a sequence of successive approximations as a linear system with initial and boundary conditions

$$\begin{aligned}
\overset{(0)}{u}(t,x) &= u^0(x) && (42)\\
\overset{(s)}{L}\overset{(s+1)}{u}_{,t} + \overset{(s)}{D}\overset{(s)}{L}\overset{(s+1)}{u}_{,x} &= \overset{(s)}{L}\overset{(s)}{b} && (43)\\
\overset{(s+1)}{u}(0,x) &= u^0(x) && (44)\\
F_j(t, \overset{(s+1)}{u}(t,0)) &= 0, \quad j = 1,\dots,m_1 && (45)\\
F_j(t, \overset{(s+1)}{u}(t,l)) &= 0, \quad j = m_1+1,\dots,m_2. && (46)
\end{aligned}$$

The existence theorem for linear system ([11]) asserts a unique solution $\overset{(s+1)}{u}$ of class C^1 for any $s = 0,1,2,\dots$ and $t \in [0,T^*]$, where T^* is defined on page 227.

6.1 Successive approximations for prolonged system

Denoting $\overset{(0)}{p} = L[u^0]\,\frac{du^0(x)}{dx}$, $\overset{(s+1)}{p} = \overset{(s)}{L}\,\overset{(s+1)}{u}_{,x}$, we consider the linear system

$$\begin{aligned}
\overset{(s+1)}{u}_{,t} &= \overset{(s)}{b} - \overset{(s)}{R}\,\overset{(s)}{D}\,\overset{(s)}{p}, && (47)\\
\overset{(s+1)}{p}_{,t} + \overset{(s)}{D}\,\overset{(s+1)}{p}_{,x} &= \overset{(s)}{L}\,\frac{\partial}{\partial x}\overset{(s)}{b} + \left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{R}\,\overset{(s)}{D}\,\overset{(s)}{p} && (48)\\
&\quad + L'(\overset{(s)}{u};\overset{(s)}{b} - \overset{(s)}{R}\,\overset{(s)}{D}\,\overset{(s)}{p})\,\overset{(s)}{R}\,\overset{(s+1)}{p} + \left(L_{,t}[\overset{(s)}{u}]\,\overset{(s)}{R} - \frac{\partial}{\partial x}\overset{(s)}{D}\right)\overset{(s+1)}{p},
\end{aligned}$$

with conditions

$$\begin{aligned}
\overset{(s+1)}{u}(0,x) &= \overset{(0)}{u}, && (49)\\
\overset{(s+1)}{p}(0,x) &= \overset{(0)}{p}, && (50)
\end{aligned}$$

$$\begin{bmatrix} \overset{(s+1)}{p_1}(t,0) \\ \vdots \\ \overset{(s+1)}{p_{m_1}}(t,0) \end{bmatrix} = \left\{ \overset{(s+1)}{\tilde{F}}_{,u}\ [\overset{(s)}{R}_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m_1}}\ [\overset{(s)}{D}_{ij}]_{\substack{j=1,\dots,m_1\\ i=1,\dots,m_1}} \right\}^{-1}_{(t,0)} \times \qquad (51)$$

$$\times \left\{ \overset{(s+1)}{\tilde{F}}_{,t} + \overset{(s+1)}{\tilde{F}}_{,u}\,\overset{(s)}{b} - \overset{(s+1)}{\tilde{F}}_{,u}\ [\overset{(s)}{R}_{ij}]_{\substack{i=1,\dots,n\\ j=m_1+1,\dots,n}}\ [\overset{(s)}{D}_{ij}]_{\substack{i=m_1+1,\dots,n\\ j=m_1+1,\dots,n}} \begin{bmatrix} \overset{(s+1)}{p_{m_1+1}} \\ \vdots \\ \overset{(s+1)}{p_n} \end{bmatrix} \right\}_{(t,0)}.$$

Similarly for the invariants $\overset{(s+1)}{p_{m_1+1}}(t,l), \dots, \overset{(s+1)}{p_{m_2}}(t,l)$.

Using induction we will demonstrate that for any $s = 0,1,\dots$ the solution of (47)-(51) exits and it is defined for each $t \in [0,T^*]$ and moreover $(\overset{(s)}{u},\overset{(s)}{p})$ stays in $B^*_\rho(u^0,p^0)$.

Assume that $(\overset{(s)}{u},\overset{(s)}{p}) \in B^*_\rho(u^0,p^0)$ for $t \in [0,T^*]$. If so, then the same estimates as on page 226 are true for $(\overset{(s+1)}{u},\overset{(s+1)}{p})$. Therefore $(\overset{(s+1)}{u},\overset{(s+1)}{p})$ is defined for $t \in [0,T^*]$ and stays in $B^*_\rho(u^0,p^0)$. Since $(\overset{(0)}{u},\overset{(0)}{p}) \in B^*_\rho(u^0,p^0)$ for any time, then $(\overset{(s)}{u},\overset{(s)}{p})_{s=0,1,\dots} \in B^*_\rho(u^0,p^0)$ for $t \in [0,T^*]$.

6.2 Uniform convergence of $\{\overset{(s)}{u}\}$

We shall show

Lemma 2. *The sequence* $\{\overset{(s)}{u}\}$ *is convergent in a Banach space* $C([0,T^*]\times\mathbb{R})$.

Proof.
We define the new unknown vector function $\overset{(s+1)}{r}(t,x) = \overset{(s)}{L}\left(\overset{(s+1)}{u} - \overset{(s)}{u}\right)$, $s=0,1,\dots$ with the initial condition $\overset{(s+1)}{r}(0,x) = [0,\dots,0]^T$. From (43) we obtain the system involving $\overset{(s+1)}{r}$, that along the characteristic curves is in the form $\frac{d}{dt}\left(\mathcal{P}_t\overset{(s+1)}{r}\right) = \overset{(s-1,s,s+1)}{h}$ where

$$\begin{aligned}\overset{(s-1,s,s+1)}{h} &= \mathcal{P}_t(\overset{(s)}{L}\overset{(s)}{b}) - \mathcal{P}_t(\overset{(s-1)}{L}\overset{(s-1)}{b}) + \mathcal{P}_t\left(L_{,t}[\overset{(s)}{u}]\overset{(s)}{R}\overset{(s+1)}{r}\right)\\ &+\mathcal{P}_t\left(L'\left(\overset{(s)}{u};\overset{(s)}{b} - \overset{(s)}{R}\overset{(s)}{D}\overset{(s)}{p}\right)\overset{(s)}{R}\overset{(s+1)}{r}\right) + \mathcal{P}_t\left(\overset{(s)}{D}\left(\frac{\partial}{\partial x}\overset{(s)}{L}\right)\overset{(s)}{R}\overset{(s+1)}{r}\right)\\ &-\mathcal{P}_t\left(\left(\overset{(s)}{L} - \overset{(s-1)}{L}\right)\overset{(s)}{u}_{,t}\right) - \mathcal{P}_t\left(\left(\overset{(s)}{D}\overset{(s)}{L} - \overset{(s-1)}{D}\overset{(s-1)}{L}\right)\overset{(s)}{u}_{,x}\right).\end{aligned}$$

Integrating each of these equations along the corresponding characteristic with respect to t from 0 to $\bar{t}$, we obtain

- if $(\bar{t},\bar{x})\in G_{piT}$, $i=1,\dots,n$ (taking into account the initial condition $\overset{(s+1)}{r_i}(0,x)=0$.)

$$\overset{(s+1)}{r_i}(\bar{t},\bar{x}) = \int_0^{\bar{t}}\left(\mathcal{P}_t\overset{(s-1,s,s+1)}{h}\right)_i dt.$$

- if $(\bar{t},\bar{x})\in G_{biT}$, $i=1,\dots,m_1$ (taking into account the boundary condition for $x=0$)

$$\overset{(s+1)}{r_i}(\bar{t},\bar{x}) = \overset{(s+1)}{r_i}(\sigma_i,0) + \int_{\sigma_i}^{\bar{t}}\left(\mathcal{P}_t\overset{(s-1,s,s+1)}{h}\right)_i dt. \tag{52}$$

- and if $(\bar{t},\bar{x})\in G_{biT}$, $i=m_1+1,\dots,m_2$ (taking into account the boundary condition for $x=l$)

$$\overset{(s+1)}{r_i}(\bar{t},\bar{x}) = \overset{(s+1)}{r_i}(\sigma_i,l) + \int_{\sigma_i}^{\bar{t}}\left(\mathcal{P}_t\overset{(s-1,s,s+1)}{h}\right)_i dt. \tag{53}$$

In the same manner as (40) we obtain:
for $i = 1, \dots, m_1$

$$\left|\overset{(s+1)}{r_i}(\sigma_i, 0)\right| \le \bar{c}(\bar{t}) \sup_{t\in[0,\bar{t}]} \| \overset{(s+1)}{r}(t,x)\|_0 + \bar{\bar{c}} \int_0^{\bar{t}} \| \overset{(s)}{r}(t,x)\|_0 dt, \tag{54}$$

for $i = m_1 + 1, \dots, m_2$

$$\left|\overset{(s+1)}{r_i}(\sigma_i, l)\right| \le \bar{c}(\bar{t}) \sup_{t\in[0,\bar{t}]} \| \overset{(s+1)}{r}(t,x)\|_0 + \bar{\bar{c}} \int_0^{\bar{t}} \| \overset{(s)}{r}(t,x)\|_0 dt, \tag{55}$$

with some nonnegative constants $\bar{c}$, $\bar{\bar{c}}$. The constant $\bar{c}$ depends on $\bar{t}$ and $\bar{c} \to 0$, when $\bar{t} \to 0$.
If $(\bar{t}, \bar{x}) \in G_{piT}$, $i = 1, \dots, n$ then we can estimate $\overset{(s+1)}{r}$(similarly to (35)):

$$|\overset{(s+1)}{r_i}(\bar{t}, \bar{x})| \le c_r \int_0^{\bar{t}} \| \overset{(s)}{r} \|_0 dt + c_r \int_0^{\bar{t}} \| \overset{(s+1)}{r} \|_0 dt, \tag{56}$$

for some nonnegative constant c_r.
From (52) and (53), for $(\bar{t}, \bar{x}) \in G_{biT}$, $i = 1, \dots, m_2$, using (54) and (55) we get:

$$|\overset{(s+1)}{r_i}(\bar{t}, \bar{x})| \le \bar{c}(\bar{t}) \sup_{t\in[0,\bar{t}]} \| \overset{(s+1)}{r}(t,x)\|_0 + (c_r + \bar{\bar{c}}) \int_0^{\bar{t}} \| \overset{(s)}{r} \|_0 dt + c_r \int_0^{\bar{t}} \| \overset{(s+1)}{r} \|_0 dt. \tag{57}$$

By inequalities (56) and (57) we deduce that for any $(\bar{t}, \bar{x}) \in [0, T] \times [0, l]$ there holds

$$\begin{aligned}\sup_{t\in[0,\bar{t}]} \| \overset{(s+1)}{r} \|_0 \quad \le \quad & \bar{c}(\bar{t}) \sup_{t\in[0,\bar{t}]} \| \overset{(s+1)}{r}(t,x)\|_0 + (c_r + \bar{\bar{c}}) \int_0^{\bar{t}} \| \overset{(s)}{r} \|_0 dt \\ & + c_r \int_0^{\bar{t}} \| \overset{(s+1)}{r} \|_0 dt.\end{aligned}$$

If $\bar{t}$ satisfies the condition

$$\bar{c}(\bar{t}) < 1, \tag{58}$$

then

$$\sup_{t\in[0,\bar{t}]} \| \overset{(s+1)}{r} \|_0 \le \frac{c_r + \bar{\bar{c}}}{1 - \bar{c}(\bar{t})} \int_0^{\bar{t}} \| \overset{(s)}{r} \|_0 dt + \frac{c_r}{1 - \bar{c}(\bar{t})} \int_0^{\bar{t}} \| \overset{(s+1)}{r} \|_0 dt. \tag{59}$$

Let $\bar{c}_r = \frac{c_r + \bar{\bar{c}}}{1 - \bar{c}(\bar{t})}$. We rewrite (59) using the quantity $\overset{(i)}{Q}(\bar{t}) = \max_{t\in[0,\bar{t}]} \| \overset{(i)}{r}(t,x)\|_0$:

$$\overset{(s+1)}{Q}(\bar{t}) \le \bar{c}_r \int_0^{\bar{t}} \overset{(s)}{Q}(t) dt + \bar{c}_r \int_0^{\bar{t}} \overset{(s+1)}{Q}(t) dt,$$

For every $t_1 \geq \bar{t}$ it is easily seen $\overset{(s+1)}{Q}(\bar{t}) \leq \bar{c}_r \int_0^{t_1} \overset{(s)}{Q}(t)dt + \bar{c}_r \int_0^{\bar{t}} \overset{(s+1)}{Q}(t)dt$. After applying Gronwall's inequality we get

$$\overset{(s+1)}{Q}(\bar{t}) \leq \bar{c}_r\, e^{\bar{c}_r\, \bar{t}} \int_0^{t_1} \overset{(s)}{Q}(t)dt \leq \bar{c}_r\, e^{\bar{c}_r\, t_1} \int_0^{t_1} \overset{(s)}{Q}(t)\, dt = c_4 \int_0^{t_1} \overset{(s)}{Q}(t)dt,$$

where $c_4 = c_r\, e^{\bar{c}_r\, T^*}$. This result holds for every $t_1 \geq \bar{t}$, hence in particular for $t_1 = \bar{t}$:

$$\overset{(s+1)}{Q}(\bar{t}) \leq c_4 \int_0^{\bar{t}} \overset{(s)}{Q}(t)dt. \tag{60}$$

Applying s-times formula (60)

$$\overset{(s+1)}{Q}(\bar{t}) \leq c_4^s \int_0^{\bar{t}} dt \int_0^{t} d\tau_1 \cdots \int_0^{\tau_{s-1}} \overset{(1)}{Q}(\tau_s) d\tau_{s-1},$$

and observing the fact that $\overset{(1)}{Q}$ is constant

$$\overset{(1)}{Q}(\bar{t}) = \max_{t\in[0,\bar{t}]} \| \overset{(0)}{r}(t,x)\|_0 \leq \max_{t\in[0,T^*]} \|L[\overset{(0)}{u}]\,(\overset{(1)}{u} - \overset{(0)}{u})\|_0 =: c_Q,$$

we conclude that

$$\overset{(s+1)}{Q}(\bar{t}) \leq \frac{(c_4\, \bar{t})^s}{s!} c_Q, \quad s = 0, 1, \ldots. \tag{61}$$

We emphasize that $\bar{t}$ has to satisfy (58). To deduce (61), which holds for any $\bar{t} \in [0, T^*]$, we take $\bar{t}$ as the initial time and become to an inequality similar to (61) for a new period of time. We continue in this fashion as long as (61) is true for $\bar{t} \in [0, T^*]$.

We are now in a position to show that $\{\overset{(s)}{u}\}$ is a Cauchy sequence in the Banach space $C([0,T^*]\times\mathbb{R})$ with the supremum norm $|||\cdot|||_0 = \max\limits_{t\in[0,T^*]} \|\cdot\|_0$. Let $k > m$. Using (61) we obtain an upper bound for the difference between any two approximations of u:

$$\begin{aligned}
&\| \overset{(k)}{u} - \overset{(m)}{u} \|_0 \leq \| \overset{(k)}{u} - \overset{(k-1)}{u} \|_0 + \cdots + \| \overset{(m+1)}{u} - \overset{(m)}{u} \|_0 \\
&= \| \overset{(k-1)}{R}\, \overset{(k)}{r} \|_0 + \cdots + \| \overset{(m)}{R}\, \overset{(m+1)}{r} \|_0 \leq c c_Q \left(\frac{(c_4\, t)^{k-1}}{(k-1)!} + \cdots + \frac{(c_4\, t)^m}{m!} \right) \\
&\leq c c_Q \frac{(c_4\, t)^m}{m!} \left(1 + \frac{c_4\, t}{m+1} + \frac{(c c_4\, t)^2}{(m+1)(m+2)} + \cdots + \frac{(c_4\, t)^{k-1-m}}{(m+1)\ldots(k-1)} \right) \\
&\leq c c_Q \frac{(c_4\, t)^m}{m!} \left(1 + \frac{c_4\, t}{1!} + \frac{(c_4\, t)^2}{2!} + \cdots + \frac{(c_4\, t)^{k-1-m}}{(k-1-m)!} \right) \leq c c_Q \frac{(c_4\, t)^m}{m!} e^{c_4\, t}.
\end{aligned}$$

Hence we deduce that the sequence $\{\overset{(s)}{u}\}$ satisfies the Cauchy criterion in the Banach space $C([0,T^*]\times[0,l])$

$$\sup_{t\in[0,T^*]} \| \overset{(k)}{u} - \overset{(m)}{u} \|_0 \le c\, c_Q \frac{(c_4\,T^*)^m}{m!}\, e^{c_4\,T^*} \longrightarrow 0, \quad \text{if} \quad m\to+\infty.$$

6.3 Equi-continuity of the sequence $\{\overset{(s)}{p}\}$

Now we will prove that for any pair of the functions (u,p) from the ball $B^*_\rho(u^0,p^0)$ that satisfy the system (21) - (22) there exists a modulus of continuity i.e. a function $\tilde{M}(\delta)$, $\tilde{M}(\delta)\to 0$ as $\delta\to 0$ such that it obeys a inequality $|p(t,x)-p(t,\bar{x})|\le \tilde{M}(\delta)$ if $|x-\bar{x}|\le\delta$ for $t\in[0,T^*]$. We show all transformations for simplicity only for p_1 (the first component of the vector function p). There are possible three cases:

- $(t,x),(t,\bar{x})\in G_{p1T}$, i.e. we consider only the initial problem;
- $(t,x),(t,\bar{x})\in G_{b1T}$, i.e. we consider only the boundary problem;
- $(t,x)\in G_{p1T},\ (t,\bar{x})\in G_{b1T}$ (or symmetrically), when we have to take into account both initial and boundary conditions.

We start from the first case with the initial condition. Let $(t,x),(t,\bar{x})\in G_{p1T}$. Considering the first equation of (22) along the characteristic curves $x=x_1(\tau;t,x)$ and integrating system with respect to τ from 0 to t, we obtain

$$\begin{aligned} p_1(t,x) \quad = \quad & p_1^0(x_1(\tau;t,x)) + \int_0^t \left(L_1[u]\frac{\partial}{\partial x} b[u]\right)(\tau,x_1(\tau;t,x))\,d\tau \\ & + \int_0^t \left(\left(\frac{\partial}{\partial x}L_1[u]\right) R[u]D[u]p\right)(\tau,x_1(\tau;t,x))\,d\tau \\ & + \int_0^t \left(L_{1,t}[u]\,R[u]p\right)(\tau,x_1(\tau;t,x))\,d\tau \\ & + \int_0^t \left(L_1'[u]\Big(u;b[u]-R[u]\,D[u]\,p\Big)\,R[u]p\right)(\tau,x_1(\tau;t,x))\,d\tau \\ & - \int_0^t \left(\left(\frac{\partial}{\partial x}\xi_1[u]\right)p_1\right)(\tau,x_1(\tau;t,x))\,d\tau. \end{aligned}$$

Our next concern will be an estimation of $|p_1(t,x)-p_1(t,\bar{x})|$.
Since $p_1^0(x_1(\tau;t,x))$ is a continuous function of x, then it is uniformly continuous on any compact set (here $x\in[0,l]$). Therefore for $|x-\bar{x}|\le\delta$ there exists a function $N_0(\delta)\to 0$ as $\delta\to 0$, such that $\left|p_1^0(x_1(\tau;t,x))-p_1^0(x_1(\tau;t,\bar{x}))\right|\le$

$N_0(\delta)$. Because of the assumption (A_1) the operator L has a bounded derivative with respect to x and hence the function $L_1(\tau, x_1(\tau; t, x)))$ satisfy the Lipschitz condition with respect to the second variable. Using this fact and also the assumption (A_4) for the operator b, we see that if $|x - \bar{x}| \leq \delta$ then

$$\begin{aligned} &\left| \left(L_1[u] \frac{\partial}{\partial x} b[u] \right) (\tau, x_1(\tau; t, x)) - \left(L_1[u] \frac{\partial}{\partial x} b[u] \right) (\tau, x_1(\tau; t, x)) \right| \\ &\leq c^2 |x_1(\tau; t, x) - x_1(\tau; t, \bar{x})| + c^2 |u_{,x}(\tau, x_1(t; t, x)) - u_{,x}(\tau, x_1(\tau; t, \bar{x}))| \\ &+ N(\delta), \end{aligned} \tag{62}$$

where $N(\delta) \to 0$ as $\delta \to 0$.
By theorem on differentiability of the solutions of ODE with respect to initial data [9] we have:

$$|x_1(\tau; t, x) - x_1(\tau; t, \bar{x})| \leq c|x - \bar{x}| < c\delta. \tag{63}$$

Let us point out that we can replace $u_{,x}$ by p on the right-hand side of (62). The operator R is bounded in the ball $B_r^1(u^0)$ and the function $R[u](t, x)$ satisfies the Lipschitz condition with respect to x. For this reason for $t \in [0, T^*]$ we obtain

$$\begin{aligned} &|u_{,x}(\tau, x_1(\tau; t, x)) - u_{,x}(\tau, x_1(\tau; t, \bar{x}))| = \\ &= |(R[u]p)(\tau, x_1(\tau; t, x)) - (R[u]p)(\tau, x_1(\tau; t, \bar{x}))| \\ &\leq c|p(\tau, x_1(\tau; t, x)) - p(\tau, x_1(\tau; t, \bar{x}))| + c^2 |x_1(\tau; t, x) - x_1(\tau; t, \bar{x})| \\ &\leq c_5 \Big(|p(\tau, x_1(\tau; t, x)) - p(\tau, x_1(\tau; t, \bar{x}))| + \delta \Big), \end{aligned} \tag{64}$$

where $c_5 = \max\{c, c^2\}$. Finally by (63) and (64) we have

$$\begin{aligned} &\left| \left(L_1[u] \frac{\partial}{\partial x} b[u] \right) (\tau, x_1(\tau; t, x)) - \left(L_1[u] \frac{\partial}{\partial x} b[u] \right) (\tau, x_1(\tau; t, \bar{x})) \right| \\ &\leq (c^3 + c_5)\delta + c^2 c_5 |p(\tau, x_1(\tau; t, x)) - p(\tau, x_1(\tau; t, \bar{x}))| + N(\delta). \end{aligned}$$

By assumptions (A_1), (A_4), (63) and (64) we get

$$\begin{aligned} &\Bigg| \left(\left(\frac{\partial}{\partial x} L_1[u] \right) R[u] D[u] p \right) (\tau, x_1(\tau; t, x)) \\ &\qquad - \left(\left(\frac{\partial}{\partial x} L_1[u] \right) R[u] D[u] p \right) (\tau, x_1(\tau; t, \bar{x})) \Bigg| \leq \\ &\leq c_6 \Big\{ N(\delta) + \delta + |p(\tau, x_1(\tau; t, \bar{x})) - p(\tau, x_1(\tau; t, \bar{x}))| \Big\}, \end{aligned}$$

for $c_6 = \max\{c^3(\rho c_5 + 1), c^3\rho, c^3\rho c_5 + 2c^3\rho\}$. In the same way we obtain

$$\left| \left(\left(\frac{\partial}{\partial x}\xi_1[u] \right) p_1 \right) (\tau, x_1(\tau; t, x)) - \left(\left(\frac{\partial}{\partial x}\xi_1[u] \right) p_1 \right) (\tau, x_1(\tau; t, \bar{x})) \right|$$
$$\leq c_7 \Big\{ N(\delta) + \delta + |p(\tau, x_1(\tau; t, \bar{x})) - p(\tau, x_1(\tau; t, \bar{x}))| \Big\},$$

where $c_7 = \max\{\rho, c^3\rho^2, c + c\rho c_5\}$.
For fixed $t \in [0, T^*]$ and for x belonging to any compact set, functions $L_t[u]$ and $L'[u]$ are uniformly continuous. Hence there exists a function $N_L(\delta) \to 0$ as $\delta \to 0$, such that there holds ($c_8 > 0$):

$$\Big| \Big(\Big(L_{1,t}[u] + L_1'(u; b[u] - R[u]D[u]p) R[u]\, D[u]\; p) \Big) R[u]p \Big) (\tau, x_1(\tau; t, x))$$
$$- \Big(\Big(L_{1,t}[u] + L_1'(u; b[u] - R[u]D[u]p) R[u]\, D[u]\; p) \Big) R[u]p \Big) (\tau, x_1(\tau; t, \bar{x})) \Big|$$
$$\leq c_8 \Big\{ \delta + N_L(\delta) + |p(\tau, x_1(\tau; t, x)) - p(\tau, x_1(\tau; t, \bar{x}))| \Big\},$$

for $c_8 = \max\{\rho(1 + c), c\rho, c^2 + c\}$. Summarizing, we see that

$$\begin{aligned} |p_1(t, x) - p_1(t, \bar{x})| \quad < \quad & N_0(\delta) + c_9 \int_0^t (N(\delta) + N_L(\delta) + \delta)\; d\tau \\ & + c_9 \int_0^t |p(\tau, x_1(\tau; t, x)) - p(\tau, x_1(\tau; t, \bar{x}))|\, d\tau, \end{aligned} \tag{65}$$

where $c_9 = \max\{c^3 + c_5, c^2 c_5, c_6, c_7, c_8\}$. The same conclusion can be drawn for every component of the vector function p.
Now we define a new function

$$M(t, \delta) = \max_{k=1,\ldots,n} \sup_{\substack{|x-\bar{x}|\leq\delta \\ \tau\leq t}} |p(\tau, x_k(\tau; t, x)) - p(\tau, x_k(\tau; t, \bar{x}))| \,. \tag{66}$$

Basis on (65) we obtain the following formula

$$M(t, \delta) \leq N_0(\delta) + t\, c_9 \Big(N(\delta) + N_L(\delta) + \delta \Big) + c_9 \int_0^t M(\tau, \delta) d\tau.$$

The next step is to apply Gronwall's inequality to the last expression

$$M(t, \delta) \leq N_0(\delta)\, e^{c_9\, t} + t \Big(N(\delta) + N_L(\delta) + \delta \Big) \left(e^{c_9\, t} - 1\right).$$

Because $N_0(\delta), N(\delta), N_L(\delta) \to 0$ as $\delta \to 0$, we conclude that, for $t \in [0, T^*]$, $M(t, \delta) \to 0$ as $\delta \to 0$. Consequently we take the definition of the function

$\tilde{M}(\delta)$ in the form $\tilde{M}_1(\delta) = N_0(\delta)\, e^{c_9\, T^*} + T^* \Big(N(\delta) + N_L(\delta) + \delta \Big)\, (e^{c_9\, T^*} - 1)$. It remains to consider the points belonging to the set in which there is a boundary problem. Let σ_1 denote time that characteristic x_1 passing through the point (t,x) enters the set $[0,T] \times [0,l]$. Similarly $\bar{\sigma}_1$ is a time that characteristic x_1 passing through the point $(t,\bar{x})$ enters the rectangle $[0,T] \times [0,l]$.
If $(t,x) \in G_{b1T}$ then taking into account the boundary condition for $x=0$ we obtain:

$$\begin{aligned} p_1(t,x) \quad = \quad & p_1(\sigma_1, 0) + \int_0^t \left(L_1[u] \frac{\partial}{\partial x} b[u] \right) (\tau, x_1(\tau; t, x))\, d\tau \\ & + \int_{\sigma_1}^t \left(\left(\frac{\partial}{\partial x} L_1[u] \right) R[u] D[u] p \right) (\tau, x_1(\tau; t, x))\, d\tau \\ & + \int_{\sigma_1}^t \left(L_{1,t}[u]\, R[u] p \right) (\tau, x_1(\tau; t, x))\, d\tau \\ & + \int_{\sigma_1}^t \Big(L_1'[u] \Big(u; b[u] - R[u]\, D[u]\, p \Big) R[u] p \Big) (\tau, x_1(\tau; t, x))\, d\tau \\ & - \int_{\sigma_1}^t \left(\left(\frac{\partial}{\partial x} \xi_1[u] \right) p_1 \right) (\tau, x_1(\tau; t, x))\, d\tau. \end{aligned}$$

Now we should estimate the difference $|p_1(t,x) - p_1(t,\bar{x})|$ for $(t,x), (t,\bar{x}) \in G_{b1T}$, but we will consider only $p_1(\sigma_1, 0) - p_1(\bar{\sigma}_1, 0)$. The rest of the elements of $|p_1(t,x) - p_1(t,\bar{x})|$ can be estimated by the right-hand side of (65). Adding and subtracting expressions in (25) (in order to use the triangle inequality and Lipschitz condition), we become to the following inequality ($i = 1, \ldots, m_1$):

$$\begin{aligned} |p_1(\sigma_1, 0) - p_1(\bar{\sigma}_1, 0)| \quad \leq \quad & c_*\, |u(\sigma_1, 0) - u(\bar{\sigma}_1, 0)| \\ & + c_* \max_{i = m_1+1, \ldots, n} |p_i(\sigma_1, 0) - p_i(\bar{\sigma}_1, 0)| \\ & + c_* \Big| F_{,u} \Big(\sigma_1, u(\sigma_1, 0) \Big) - F_{,u} \Big(\bar{\sigma}_1, u(\bar{\sigma}_1, 0) \Big) \Big| \\ & + c_* \Big| F_{,t} \Big(\sigma_1, u(\sigma_1, 0) \Big) - F_{,t} \Big(\bar{\sigma}_1, u(\bar{\sigma}_1, 0) \Big) \Big|, \end{aligned}$$

where $F_{,u} := [F_{j,u_i}]_{\substack{j=1,\ldots,m_1 \\ i=1,\ldots,n}}$, $F_{,t} := [F_{1,t}, \ldots, F_{m_1,t}]^T$. The constant c_* is the maximum of the following quantities: $\sup\limits_{[0,T]\times[0,l]} \left| \Big\{ F_{,u} [R_{ij}]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m_1}} [D_{ij}]_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,m_1}} \Big\}^{-1} \right|$, $\max\limits_{[0,T]\times[0,l]} \Big\{ |D|, |R|, |b| \Big\}$, $\max\limits_{[0,T]} \Big\{ |F_{,u}|, |F_{,t}| \Big\}$, $\sup\limits_{[0,T]\times[0,l]} \left| F_{,u} [R_{ij}]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m_1}} \right|$, $\sup\limits_{[0,T]\times[0,l]} \left| F_{,u} [D_{ij}]_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,m_1}} \right|$, $\sup\limits_{[0,T]\times[0,l]} \left| [R_{ij}]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m_1}} [D_{ij}]_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,m_1}} \right|$.

For the fixed t the functions $F_{,u}$, $F_{,t}$, u are continuous with respect to x ($x \in [0, l]$). Therefore all this functions are uniformly continuous and there exists a function $\tilde{N}(\delta)$, $\tilde{N}(\delta) \to 0$ as $\delta \to 0$, such that

- $|u(\sigma_1, 0) - u(\bar{\sigma}_1, 0)| \leq \tilde{N}(\delta)$,
- $\Big|F_{,t}\Big(\sigma_1, u(\sigma_1, 0)\Big) - F_{,t}\Big(\bar{\sigma}_1, u(\bar{\sigma}_1, 0)\Big)\Big| \leq \tilde{N}(\delta)$,
- $\Big|F_{,u}\Big(\sigma_1, u(\sigma_1, 0)\Big) - F_{,u}\Big(\bar{\sigma}_1, u(\bar{\sigma}_1, 0)\Big)\Big| \leq \tilde{N}(\delta)$.

Let us remind that the points $(\sigma_1, 0)$ and $(\bar{\sigma}_1, 0)$ belong to the G_{piT} for $i = m_1 + 1, \ldots, n$ i.e. they are in the sets where Cauchy condition is considered. Hence by reasoning for the first part of that paragraph there exists a function $\tilde{\tilde{N}}(\delta)$, $\tilde{\tilde{N}}(\delta) \to 0$ as $\delta \to 0$, such that $\max_{i=m_1+1,\ldots,n} |p_i(\sigma_1, 0) - p_i(\bar{\sigma}_1, 0)| \leq \tilde{\tilde{N}}(\delta)$. Finally we obtain $|p_1(\sigma_1, 0) - p_1(\bar{\sigma}_1, 0)| \leq 3c_* \tilde{N}(\delta) + c_* \tilde{\tilde{N}}(\delta)$ and for $(t, x), (t, \bar{x}) \in G_{b1T}$:

$$\begin{aligned} |p_1(t,x) - p_1(t,\bar{x})| \quad < \quad & c_{10}\tilde{N}(\delta) + c_{10}\tilde{\tilde{N}}(\delta) + c_{10}\int_0^t \left(N(\delta) + N_L(\delta) + \delta\right) d\tau \\ & + c_{10}\int_0^t |p(\tau, x_1(\tau; t, x)) - p(\tau, x_1(\tau; t, \bar{x}))| \, d\tau, \end{aligned} \tag{67}$$

where $c_{10} = \max\{3c_*, c^3 + c_5, c^2 c_5, c_6, c_7, c_8\}$.
From (67), for the function $M(t, \delta)$ (given by (66)) there holds

$$M(t,\delta) \leq c_{10}\tilde{N}(\delta) + c_{10}\tilde{\tilde{N}}(\delta) + t\, c_{10}\Big(N(\delta) + N_L(\delta) + \delta\Big) + c_{10}\int_0^t M(\tau, \delta) d\tau.$$

By Gronwall's lemma, we have

$$M(t,\delta) \leq \Big(c_{10}\tilde{N}(\delta) + c_{10}\tilde{\tilde{N}}(\delta)\Big)e^{c_{10}\, t} + t\Big(N(\delta) + N_L(\delta) + \delta\Big)\left(e^{c_{10}\, t} - 1\right).$$

Now we choose

$$\tilde{M}_2(\delta) = \Big(c_{10}\tilde{N}(\delta) + c_{10}\tilde{\tilde{N}}(\delta)\Big)e^{c_{10}\, T^*} + T^*\Big(N(\delta) + N_L(\delta) + \delta\Big)\left(e^{c_{10}\, T^*} - 1\right).$$

The same conclusion can be drawn for $(t, x), (t, \bar{x}) \in G_{biT}$, $i = m_1 + 1, \ldots, m_2$. Basis on the presented reasoning we conclude that the functions of the sequence $\{\overset{(s_k)}{p}\}$ which satisfy (47) are equi-continuous with respect to x for the fixed $t \in [0, T^*]$. We will not consider the third case, i.e. when $(t, x) \in G_{piT}$ and $(t, \bar{x}) \in G_{biT}$ for $i = 1, \ldots, m_1$ or $(t, x) \in G_{piT}$ and $(t, \bar{x}) \in G_{biT}$ for $i = m_1 + 1, \ldots, m_2$. Below we prove a lemma, which shows, that as a modulus of continuity we can take a sum of the two functions: $\tilde{M}_1(\delta)$ i $\tilde{M}_2(\delta)$.

Lemma 3. *If a sequence of functions $\{f_n(x)\}$ is equi-continuous on $[a,b]$ and on $[b,c]$, then it is also equi-continuous on $[a,c]$.*

Proof. Let

$$M_1(\delta) = \sup_{\substack{|x-y|<\delta \\ x,y\in[a,b]}} |f_n(x) - f_n(y)|, \quad M_2(\delta) = \sup_{\substack{|x-y|<\delta \\ x,y\in[b,c]}} |f_n(x) - f_n(y)|.$$

It is easy to see that

$$\begin{aligned} \sup_{\substack{|x-y|<\delta \\ x\in[a,b] \\ y\in[b,c]}} |f_n(x) - f_n(y)| &\le \sup_{\substack{|x-b|<\delta \\ x\in[a,b]}} |f_n(x) - f_n(b)| + \sup_{\substack{|b-y|<\delta \\ y\in[b,c]}} |f_n(b) - f_n(y)| \\ &\le M_1(\delta) + M_2(\delta). \end{aligned}$$

□

By the Arzela-Ascoli theorem, if functions of a sequence are equi-bounded and equi-continuous then there exists a uniformly convergent subsequence. Therefore some subsequence $\{\overset{(s_k)}{p}\}$ converges uniformly on $[0,l]$ to the continuous function $p(t,x)$ for fixed $t \in [0,T^*]$. We have proved that the sequence $\{\overset{(s)}{u}\}$ converges uniformly to the continuous function $u(t,x)$. Hence we are able to consider a subsequence $\{\overset{(s_k)}{u}\}$, which obviously is uniformly converged to $u(t,x)$. Under the assumptions for the operator R we have $\|R[\overset{(s_k)}{u}] - R[u]\|_0 \le c\|\overset{(s_k)}{u} - u\|_0$. Consequently, for fixed $t \in [0,T^*]$, $R[\overset{(s_k)}{u}]$ converges uniformly to $R[u]$ and $R[\overset{(s_k)}{u}]\overset{(s_k)}{p}$ converges uniformly to the continuous function $R[u]\,p$. Note that $\overset{(s_k)}{u_x} = R[\overset{(s_k)}{u}]\overset{(s_k)}{p}$. We conclude that the function $u(t,x)$ is continuously differentiable for $t \in [0,T^*]$, $x \in [0,l]$ and $u_x = R[u]\,p$. Since the derivative of $u(t,x)$ is unique, then every uniformly convergent subsequence of $\{\overset{(s)}{p}\}$ converges to $p(t,x)$. The sequence $\{\overset{(s)}{p}\}$ has the only one limit point.

7 Proof of $(A_1) - (A_5)$ for system (1)-(6)

For the unknown vector function $u = [N_a, n_i, V_i, T_e]^T$, the total current density $I(t)$ is, in fact, given by the functional

$$I[u] = \left(\int_0^l \frac{\nu_{eff}}{e\,n_i}dx\right)^{-1} \cdot \left[\frac{e}{m_i}U_0 + \int_0^l \left(\nu_{eff}V_i + \frac{1}{n_i}\frac{\partial}{\partial x}\left(\frac{kT_e}{m_i}n_i\right)\right)dx\right].$$

We write $I(t)$ to emphasize that values of $I[u]$ do not depend on x, but they can depend on t, because the solution usually depends on t.
V_e in Eq. (4) denotes the electron velocity component in the axial direction. It is given by $V_e = V_i - \frac{I[u]}{n_i e}$. The characteristics of the system (1) - (4) have the following slopes: $\xi_1 = V_a,\ \xi_2 = V_i - \sqrt{\frac{5kT_e}{3m_i}},\ \xi_3 = V_i + \sqrt{\frac{5kT_e}{3m_i}},\ \xi_4 = V_e$ and the appropriate left eigenvectors are

$$\begin{aligned}
L_1 &= [1,0,0,0],\\
L_2 &= \left[0,\ \frac{T_e}{n_i} - \frac{m_i}{k\,e\,n_i^2}\sqrt{\frac{5kT_e}{3m_i}}\,I[u]\ ,\ \ \frac{m_i}{k\,e\,n_i}\,I[u] - \frac{m_i}{k}\sqrt{\frac{5kT_e}{3m_i}},\ \ 1\right],\\
L_3 &= \left[0,\ \frac{T_e}{n_i} + \frac{m_i}{k\,e\,n_i^2}\sqrt{\frac{5kT_e}{3m_i}}\,I[u]\ ,\ \ \frac{m_i}{k\,e\,n_i}\,I[u] + \frac{m_i}{k}\sqrt{\frac{5kT_e}{3m_i}},\ \ 1\right],\\
L_4 &= \left[0,1,0,-\frac{3}{2}\frac{n_i}{T_e}\right].
\end{aligned}$$

The matrix R of the right eigenvectors (columns) and the vector b on the right hand side of the system are in the form:

$$R[u] = \begin{bmatrix}
1 & 0 & 0 & 0\\
0 & \frac{kn_i}{2m_i}\frac{1}{\sqrt{\frac{5kT_e}{3m_i}}\,W_1} & \frac{kn_i}{2m_i}\frac{1}{\sqrt{\frac{5kT_e}{3m_i}}\,W_2} & \frac{2kT_e}{3m_i\ W_1W_2}\\
0 & -\frac{k}{2m_i}\frac{1}{W_1} & \frac{k}{2m_i}\frac{1}{W_2} & \frac{-2kT_e\,\frac{I[u]}{en_i}}{3n_im_iW_1W_2}\\
0 & \frac{kT_e}{3m_i}\frac{1}{\sqrt{\frac{5kT_e}{3m_i}}\,W_1} & \frac{kT_e}{3m_i}\frac{1}{\sqrt{\frac{5kT_e}{3m_i}}\,W_2} & \frac{-2T_e}{3n_i}\frac{\left(\sqrt{\frac{kT_e}{m_i}}\right)^2-\left(\frac{I[u]}{en_i}\right)^2}{W_1W_2}
\end{bmatrix},$$

where $W_1 := \sqrt{\frac{5kT_e}{3m_i}} - \frac{I[u]}{en_i}$, $W_2 := \sqrt{\frac{5kT_e}{3m_i}} + \frac{I[u]}{en_i}$ and

$$b[u] = \begin{bmatrix}
-\beta N_a n_i\\
\beta N_a n_i\\
\nu_{eff}\left(\frac{\mathbf{I[u]}}{en_i} - V_i\right) + \beta N_a(V_a - V_i)\\
\frac{2}{3}Q
\end{bmatrix}.$$

Here Q is given by (6).
We assume that the initial data for the unknown functions N_a, n_i, V_i, T_e are chosen in that way that $\sqrt{\frac{5kT_e}{3m_i}} - \frac{I[u]}{en_i} \neq 0$. Otherwise the eigenvectors of A become linearly dependent. Because of the physical sense the functions N_a and n_i should be positive. And also the assumption $n_e = n_i$ can made singularity since n_e appears in denominator. Below we show that for positive initial data

both functions will be also positive.
Let $u^0 = u(0,x) = [N_{a0}(x), n_{i0}(x), V_{i0}(x), T_{e0}(x)]^T$, $u \in B^1_r(u^0) \in C^1([0,l])$.
We can assume $N_a(t,0) = N^*_a(t) \geq \alpha > 0$, $n_i(t,0) \geq \alpha > 0$, $N_{a0}(x) \geq \alpha > 0$.
From (1), we have on the curves $\frac{dx}{dt} = V_a$, that $\frac{d \ln N_a}{dt} = -\beta n_e - V_{a,x}$. Then $\left\| \frac{d \ln N_a}{dt} \right\|_0 \leq \beta_{max}(r + \|n_{i0}\|_0) + \|V_{a,x}\|_0$. Since $\left\| \frac{d \ln N_a}{dt} \right\|_0$ is bounded, we see that $\| \ln N_a\|_0$ is also bounded. Therefore if there exists classical, bounded solution of the scheme (1) - (4) and $N_{a0} \geq \alpha > 0$ then the function N_a is positive and bounded for $t \geq 0$. The same conclusion can be drawn for the function n_i. On the curves $\frac{dx}{dt} = V_i$, which cover all the domain, there is (from (2)) $\frac{d \ln n_i}{dt} = \beta N_a - V_{i,x}$. Hence $\frac{d \ln n_i}{dt} \geq -\|V_{i,x}\|_0$, and $n_i(t,x) \geq n_{i0} e^{-\int_0^t \|V_{i,x}\|_0 d\tau}$.
As $n_{i0} \geq \alpha > 0$ we see that n_i is the positive and bounded function for $t \geq 0$.

Now we will proof that all assumptions $(A_1) - (A_5)$ are fulfilled for the system (1)-(6). Let us notice, that $\xi_4 = V_e$; L_2; L_3 and the right-hand side of the system (see Eqs.(3) and (4)) depend on the total current density $I[u]$ and thus depend functionally on the solution.
In the coefficients of equations (3) and (4), the functional $I[u]$ is multiplied by the functions continuously differentiable. Moreover $\nu_{eff}(x) > 0$ and the function n_i is bounded from above, because it belongs to the ball $B^1_r(u^0)$. Hence we only need to proof that assumptions $(A_1) - (A_5)$ hold for $I[u]$.
To shorten notation we will write ν instead of ν_{eff}. We will also ignore the constant coefficients $\frac{1}{e}$, $\frac{k}{m_i}$.
For $u \in B^1_r(u^0)$ and $t \in [0,T]$ we have
Assumption (A_1) : Functions, that are in $I[u]$, i.e. u and ν_{eff}, are continuously differentiable and $u \in B^1_r(u^0)$, $\nu_{eff} > 0$. Hence $I[u]$ is bounded.
Assumption (A_2) : Let $h = [h_1, h_2, h_3, h_4]^T$, $h \in X_1$ and correspond with the functions $u_1 = N_a$, $u_2 = n_i$, $u_3 = V_i$, $u_4 = T_e$. We define

$$K_1[u] = \int_0^l \frac{\nu}{n_i} dx,\ K_2[u] = \int_0^l \nu\, V_i dx,\ K_3[u] = \int_0^l \frac{1}{n_i} \frac{\partial(n_i T_e)}{\partial x} dx.$$

Thus $I = \frac{1}{K_1}\left(\frac{e}{m_i} U_0 + K_2 + K_3\right)$. It is well known a rule of differentiating a quotient (if a numerator and a denominator are differentiable). Therefore we only calculate the Fréchet derivative for the functionals K_1, K_2, K_3:

- $K_1'[u]h = -\int_0^l \frac{\nu}{n_i^2} h_2\, dx$ and $\|K_1'[u]h\|_0 \leq a_1 \|h_2\|_0 \leq a_1 \|h\|_0$, where $a_1 = \int_0^l \frac{\nu}{n_i^2}\, dx$;
- $K_2'[u]h = \int_0^l \nu\, h_3\, dx$ and $\|K_2'[u]h\|_0 \leq a_2 \|h_3\|_0 \leq a_2 \|h\|_0$, where $a_2 = \int_0^l \nu\, dx$;
- $K_3'[u]h = h_4(l) - h_4(0) + \int_0^l \frac{1}{n_i} \frac{\partial n_i}{\partial x} h_4\, dx - \int_0^l \frac{T_e}{n_i^2} \frac{\partial n_i}{\partial x} h_2\, dx + \int_0^l \frac{T_e}{n_i} \frac{dh_2}{dx}\, dx.$

It is enough to take $h \in X_0$ because thanks to integrating by parts we can eliminate the derivative $\frac{dh_2}{dx}$:

$$\begin{aligned} K_3'[u]h &= h_4(l) - h_4(0) + \int_0^l \frac{1}{n_i}\frac{\partial n_i}{\partial x} h_4\,dx - \int_0^l \frac{T_e}{n_i^2}\frac{\partial n_i}{\partial x} h_2\,dx \\ &\quad + \left[h_2(x)\,\frac{T_e(t,x)}{n_i(t,x)}\right]_{x=0}^{x=l} - \int_0^l h_2 \frac{\partial}{\partial x}\left(\frac{T_e}{n_i}\right) dx \\ &= h_4(l) - h_4(0) + \left[h_2(x)\,\frac{T_e(t,x)}{n_i(t,x)}\right]_{x=0}^{x=l} + \int_0^l \frac{1}{n_i}\frac{\partial n_i}{\partial x} h_4\,dx \\ &\quad - \int_0^l \frac{h_2}{n_i}\frac{\partial T_e}{\partial x}\,dx. \end{aligned}$$

This gives

$$\begin{aligned} \|K_3'[u]h\|_0 &\le 2\|h_4\|_0 + \|h_4\|_0 \cdot \int_0^l \left|\frac{1}{n_i}\frac{\partial n_i}{\partial x}\right| dx + \|h_2\|_0 \cdot \int_0^l \left|\frac{1}{n_i}\frac{\partial T_e}{\partial x}\right| dx \\ &\quad + \|h_2\|_0 \cdot 2 \left\|\frac{T_e}{n_i}\right\|_0 \le a_3\,\|h\|_0, \end{aligned}$$

where $a_3 = 2\max\left\{2 + \int_0^l \left|\frac{1}{n_e}\frac{\partial n_e}{\partial x}\right| dx \;\; ; \;\; \int_0^l \left|\frac{1}{n_e}\frac{\partial T}{\partial x}\right| dx + 2\left\|\frac{T}{n_e}\right\|_0\right\}$.

Finally we obtain $\|I'[u]h\|_0 \le a\,\|h\|_0$ for the constant a, which is expressed by a_1, a_2, a_3, $\frac{e}{m_i}U_0$.

Assumption (A_3) : As $I[u]$ does not depend explicit on t we have $\left(\frac{\partial}{\partial t} I\right)[u] = 0$.

Assumption (A_4) : If $K[u](x) = g(x,u)$, where g is continuously differentiable function with respect to x and u, then the assumption (A_4) is immediately satisfied (taking $N(\delta) = 0$). The operators in the coefficients of the system are products of the differentiable functions and the functional $I[u]$, e.g. $\frac{1}{n_e}\,I[u]$. Since $I[u]$ is bounded and $\frac{\partial}{\partial x}\,(I[u]) = 0$, assumption (A_4) holds.

Assumption (A_5) : We can check if the assumption (A_5) holds by estimating $\|I[u] - I[\bar{u}]\|_0$. In the other hand it is easily seen that the Fréchet derivative is bounded and finally the Lipschitz condition (A_5) holds. Since for $h \in X_0$ we have $\|I'[u]h\|_0 \le c\,\|h\|_0$, the norm of the Fréchet derivative is bounded by c. It implies $\|I[u] - I[\bar{u}]\|_0 \le c\,\|u - \bar{u}\|_0$ for $u \in B_r^1(u^0)$.

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Małgorzata ZDANOWICZ,
Institute of Mathematics,
University of Białystok,
K.Ciokowskiego 1M, 15-245 Białystok, Poland.
Email: mzdan@math.uwb.edu.pl

Zbigniew PERADZYŃSKI,
Faculty of Mathematics, Informatics and Mechanics,
University of Warsaw,
Banacha 2, 02-097 Warsaw, Poland.