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Properties of a new integral operator

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Abstract

In this paper, we derive sufficient conditions for the univalence, starlikeness, convexity and some other properties in the class $N(\rho)$, for a new integral operator defined on the space of normalized analytic functions in the open unit disk.

1 Introduction

Let $\mathcal{A}$ be the class of functions which are analytic in the open unit disk $U = \{z : |z| < 1\}$ given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U. \tag{1.1}$$

Consider S the subclass of $\mathcal{A}$ consisting of univalent functions. We denote by $S^*(\alpha)$ the class of starlike univalent functions of order α $(0 \leq \alpha < 1)$,

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{zf^{'}(z)}{f(z)}\right] > \alpha,\ z \in U \right\}.$$

By $K(\alpha)$ we denote a subclass of $\mathcal{A}$ consisting of convex univalent functions of order α $(0 \leq \alpha < 1)$ defined as

$$K(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{zf^{''}(z)}{f'(z)} + 1\right] > \alpha,\ z \in U \right\}.$$

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Clearly, we have
(i) $S^*(0) = S^*$ the class of all starlike functions with respect to the origin;
(ii)$K(0) = K$ the class of all convex functions;
(iii) $K \subset S^* \subset S, \quad K(\alpha) \subset S^*(\alpha), \quad K(\alpha) \subset K$ and $S^*(\alpha) \subset S^*$.

A function $f \in \mathcal{A}$ is said to be in the class R_λ if and only if
$\operatorname{Re}\left[f^{'}(z)\right] > \lambda$, for some λ, $0 \leq \lambda < 1$.

Recently, Frasin and Jahangiri [4] define the family $B(\mu, \lambda), \mu \geq 0, 0 \leq \lambda < 1$ consisting of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f^{'}(z) \left[\frac{z}{f(z)}\right]^{\mu} - 1 \right| < 1 - \lambda, \tag{1.2}$$

for all $z \in U$.
It is obvious that: (i) $B(0, \lambda) = R_\lambda$; (ii) $B(1, \lambda) = S^*(\lambda)$;
(iii) $B(2, \lambda) = B(\lambda)$ (see Frasin and Darus [5]);
(iv) $B(2, 0) = S$ (see Ozaki and Nunokawa [3]).

Let $N(\rho)$ be the subclass of $\mathcal{A}$ that contains all the functions f which satisfy the inequality

$$\operatorname{Re}\left[\frac{zf^{''}(z)}{f'(z)} + 1\right] < \rho, \quad \rho > 1, z \in U.$$

Uralegaddi, Ganigi and Sarangi in [11] and Owa and Srivastava in [7] introduced and studied the class $N(\rho)$.

In the present paper, we introduce a new integral operator

$$J_\alpha : \mathcal{A}\text{x}\mathcal{A} \to \mathcal{A}$$

defined by:

$$J_\alpha(f, g)(z) = \int_0^z \left[\frac{e^{f(t)}}{g^{'}(t)}\right]^{\alpha} dt, \tag{1.3}$$

where parameter α is a complex number, with $\operatorname{Re}\alpha \geq 1$.

In this paper our purpose is to obtain univalence conditions, starlikeness properties, the order of convexity for the integral operator abovementioned and to show that the operator $J_\alpha(f, g)(z)$ is in the class $N(\rho)$, by using functions from the class $B(\mu, \lambda)$. Recently, various types of integral operators were studied by different authors (see [10, 2]), and some of them motivated us to come up with the integral operator defined in (1.3).

In the proof of our main results, we need to recall here the following:

Theorem 1.1. *(Becker [1]) If the function f is regular in the unit disk U, $f(z) = z + a_2 z^2 + \dots$ and*

$$(1 - |z|^2) \cdot \left| \frac{z f''(z)}{f'(z)} \right| \leq 1 \tag{1.4}$$

for all $z \in U$, then the function f is univalent in U.

Lemma 1.1. *(General Schwarz Lemma [6]) Let f be regular function in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in z=0 one zero with multiply bigger than m, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in U_R. \tag{1.5}$$

The equality case hold only if $f(z) = e^{i\theta} \frac{M}{R^m} z^m$, where θ is constant.

Lemma 1.2. *[9] Let the functions p and q be analytic in U with*

$$p(0) = q(0) = 0,$$

and let δ be a real number. If the function q maps the unit disk U onto a region which is starlike with respect to the origin, the inequality

$$Re \left[\frac{p'(z)}{q'(z)} \right] > \delta, \text{ for all } z \in U$$

implies that

$$Re \left[\frac{p(z)}{q(z)} \right] > \delta, \text{ for all } z \in U.$$

2 Main results

The univalence condition for the operator $I_\alpha(f, g)$ defined in (1.3) is proved in the next theorem, by using the Becker univalence criterion.

Theorem 2.1. *Let α be a complex number, with $\mathrm{Re}\alpha \geq 1$, $f \in B(\mu, \lambda)$ and $g \in \mathcal{A}$. Suppose also that positive real numbers M $(M \geq 1)$ and N $(N \geq 1)$ are so constrained that*

$$|f(z)| < M \text{ and } \left| \frac{g''(z)}{g'(z)} \right| \leq N, \quad z \in U. \tag{2.1}$$

If

$$|\alpha| \leq \frac{3\sqrt{3}}{2[(2 - \lambda) M^\mu + N]}, \tag{2.2}$$

then the function $J_\alpha(f, g)$ is in the class S.

Proof. Let the function h be defined by

$$h(z) := J_\alpha(f,g)(z), \quad z \in U. \tag{2.3}$$

Obviously h is regular in U and $h(0) = h^{'}(0) - 1 = 0$. From (2.3) we obtain

$$\frac{zh^{''}(z)}{h^{'}(z)} = \alpha\left[zf^{'}(z) - \frac{zg^{''}(z)}{g^{'}(z)}\right]. \tag{2.4}$$

Hence, we get

$$(1-|z|^2)\cdot\left|\frac{zh^{''}(z)}{h^{'}(z)}\right| \le (1-|z|^2)\cdot|z|\cdot|\alpha|\left[\left|f^{'}(z)\right| + \left|\frac{g^{''}(z)}{g^{'}(z)}\right|\right] \tag{2.5}$$

$$\le (1-|z|^2)\cdot|z|\cdot|\alpha|\left[\left(\left|f^{'}(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| + 1\right)\left|\frac{f(z)}{z}\right|^{\mu} + \left|\frac{g^{''}(z)}{g^{'}(z)}\right|\right]. \tag{2.6}$$

By using the hypothesis of the theorem and applying the General Schwarz Lemma, we have

$$(1-|z|^2)\cdot\left|\frac{zh^{''}(z)}{h^{'}(z)}\right| \le (1-|z|^2)\cdot|z|\cdot|\alpha|\left[(2-\lambda)M^{\mu} + N\right]. \tag{2.7}$$

Considering the function

$$t : [0,1) \to R,$$
$$t(x) = x(1-x^2),\ x = |z|,$$

we find that

$$t(x) \le \frac{2}{3\sqrt{3}},\ \text{for all } x \in [0,1). \tag{2.8}$$

From (2.7), (2.8) and (2.6) we obtain

$$(1-|z|^2)\cdot\left|\frac{zh^{''}(z)}{h^{'}(z)}\right| \le \frac{2|\alpha|}{3\sqrt{3}}\left[(2-\lambda)M^{\mu} + N\right] \le 1. \tag{2.9}$$

Finally, by applying Theorem 1.1 in (2.9) we yield that the function $J_\alpha(f,g)$ is in the class S. $\square$

In the following theorem we give sufficient conditions such that the integral operator $J_\alpha(f,g) \in S^*$.

Theorem 2.2. *Let α be a complex number, with* $\operatorname{Re}\alpha \geq 1$, $f \in B(\mu, \lambda)$ *and* $g \in \mathcal{A}$. *Suppose also that positive real number M ($M \geq 1$) is so constrained that*

$$|f(z)| < M \text{ and } \left|\frac{zg^{''}(z)}{g^{'}(z)}\right| < 1, \quad z \in U. \tag{2.10}$$

If

$$|\alpha| \leq \frac{1}{(2-\lambda)M^{\mu}+1}, \tag{2.11}$$

then the function $J_\alpha(f, g)$ is in the class S^*.

Proof. For the function h be given by (2.3) we obtain

$$\frac{zh^{'}(z)}{h(z)} = \frac{\frac{ze^{\alpha f(z)}}{\left[g^{'}(z)\right]^{\alpha}}}{\int_0^z \left[\frac{e^{f(t)}}{g^{'}(t)}\right]^{\alpha} dt}. \tag{2.12}$$

Setting

$$p(z) = zh^{'}(z) \text{ and } q(z) = h(z),$$

we find that $p(0) = q(0) = 0$, and q satisfies the starlikeness condition of Lemma 1.2. Since,

$$\frac{p^{'}(z)}{q^{'}(z)} = 1 + \alpha \left[zf^{'}(z) - \frac{zg^{''}(z)}{g^{'}(z)}\right]$$

we obtain

$$\left|\frac{p^{'}(z)}{q^{'}(z)} - 1\right| \leq \cdot|\alpha| \left[\left(\left|f^{'}(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| + 1\right)\frac{|f(z)|^{\mu}}{|z|^{\mu-1}} + \left|\frac{zg^{''}(z)}{g^{'}(z)}\right|\right]. \tag{2.13}$$

Also, since $|f(z)| < M$, $z \in U$, applying the Schwarz Lemma, we have

$$\left|\frac{f(z)}{z}\right| \leq M, \text{ for all } z \in U. \tag{2.14}$$

By using the hypothesis of the Theorem and replacing (2.14) in inequation (2.13), we obtain

$$\left|\frac{p^{'}(z)}{q^{'}(z)} - 1\right| \leq |\alpha| \cdot \left[\left(\left|f^{'}(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| + 1\right)M^{\mu}|z| + 1\right] \leq |\alpha|[1+(2-\lambda)\cdot M^{\mu}] \leq 1.$$

Thus, we have

$$Re\left[\frac{p^{'}(z)}{q^{'}(z)}\right] > 0, \quad z \in U \tag{2.15}$$

and, applying Lemma 1.2, we find that

$$Re\left[\frac{p(z)}{q(z)}\right] > 0, \quad z \in U. \tag{2.16}$$

This completes the proof of the theorem. □

Letting $\mu = 1$ in Theorem 2.2, we have

Corollary 2.1. *Let α be a complex number, with* $\mathrm{Re}\alpha \geq 1$, *$f \in S^*(\lambda)$ and $g \in \mathcal{A}$. Suppose also that positive real number M, $M \geq 1$ is so constrained that*

$$|f(z)| < M \ \textit{and} \ \left|\frac{zg''(z)}{g'(z)}\right| < 1, \quad z \in U.$$

If

$$|\alpha| \leq \frac{1}{1+(2-\lambda)M},$$

then the function $J_\alpha(f,g)$ is in the class S^*.

Letting $\lambda = 0$ in Corollary 2.1, we obtain

Corollary 2.2. *Let α be a complex number, with* $\mathrm{Re}\alpha \geq 1$, *$f \in S^*$ and $g \in \mathcal{A}$. Suppose also that positive real number M, $M \geq 1$ is so constrained that*

$$|f(z)| < M \ \textit{and} \ \left|\frac{zg''(z)}{g'(z)}\right| < 1, \quad z \in U.$$

If

$$|\alpha| \leq \frac{1}{1+2M},$$

then the function $J_\alpha(f,g)$ is in the class S^*.

Theorem 2.3. *Let α be a complex number, with* $\mathrm{Re}\alpha \geq 1$, *$f \in B(\mu,\lambda)$ and $g \in \mathcal{A}$. Suppose also that positive real numbers M $(M \geq 1)$ and N $(N \geq 1)$ are so constrained that*

$$|f(z)| < M \ \textit{and} \ \left|\frac{g''(z)}{g'(z)}\right| < N, \quad z \in U.$$

Then the function $J_\alpha(f,g)$ is in the class $K(\delta)$, where

$$\delta = 1 - |\alpha|[N + (2-\lambda)\cdot M^\mu] \ \textit{and} \ 0 < |\alpha|[N + (2-\lambda)\cdot M^\mu] \leq 1.$$

Proof. By letting the function h defined in (2.3), from equation (2.18) we find that

$$\left|\frac{zh^{''}(z)}{h^{'}(z)}\right| \le |z|\cdot|\alpha|\left[\left|f^{'}(z)\right| + \left|\frac{g^{''}(z)}{g^{'}(z)}\right|\right]$$
$$\le |z|\cdot|\alpha|\left[\left(\left|f^{'}(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| + 1\right)\left|\frac{f(z)}{z}\right|^{\mu} + \left|\frac{g^{''}(z)}{g^{'}(z)}\right|\right]. \quad (2.17)$$

From the hypothesis and applying the Schwarz Lemma in inequation (2.17), we obtain

$$\left|\frac{zh^{''}(z)}{h^{'}(z)}\right| \le |\alpha|[N + (2-\lambda)\cdot M^{\mu}] = 1 - \delta.$$

This evidently completes the proof. □

Letting $\mu = 1$ in Theorem 2.3, we have

Corollary 2.3. *Let α be a complex number, with* $\mathrm{Re}\alpha \ge 1$, $f \in S^*(\lambda)$ *and* $g \in \mathcal{A}$. *Suppose also that positive real numbers M $(M \ge 1)$ and N $(N \ge 1)$ are so constrained that*

$$|f(z)| < M \ \ and \ \left|\frac{g^{''}(z)}{g^{'}(z)}\right| < N, \quad z \in U.$$

Then the function $J_\alpha(f,g)$ is in the class $K(\delta)$, where

$$\delta = 1 - |\alpha|[N + (2-\lambda)M] \ and \ 0 < |\alpha|[N + (2-\lambda)M] \le 1.$$

Letting $\delta = \lambda = 0$ in Corollary 2.3, we obtain

Corollary 2.4. *Let α be a complex number*, *with* $\mathrm{Re}\alpha \ge 1$, $f \in S^*$ *and* $g \in \mathcal{A}$. *Suppose also that positive real numbers M $(M \ge 1)$ and N $(N \ge 1)$ are so constrained that*

$$|f(z)| < M \ \ and \ \left|\frac{g^{''}(z)}{g^{'}(z)}\right| < N, \quad z \in U.$$

Then the function $J_\alpha(f,g)$ is in the class K, where

$$|\alpha| = \frac{1}{2M + N}.$$

Theorem 2.4. *Let the functions $f, g \in \mathcal{A}$, with f in the class $B(\mu, \lambda)$, $\mu \geq 0, 0 \leq \lambda < 1$, and α a complex number, with $\mathrm{Re}\alpha \geq 1$. If $|f(z)| < M$, for M a positive real number, $M \geq 1$, $z \in U$ and $\left|\frac{g''(z)}{g'(z)}\right| < 1$, then the integral operator $J_\alpha(f, g)$ defined by (1.3) is in the class $N(\rho)$, where*

$$\rho = |\alpha| \left[1 + (2 - \lambda) M^\mu\right] + 1.$$

Proof. From (2.4) we obtain that

$$\frac{zJ_\alpha''(f,g)(z)}{J_\alpha'(f,g)(z)} = \alpha z \left[f'(z) - \frac{g''(z)}{g'(z)}\right]$$

So,

$$\begin{aligned}
\mathrm{Re}\left[\frac{zJ_\alpha''(f,g)(z)}{J_\alpha'(f,g)(z)} + 1\right] &= \mathrm{Re}\left[\alpha z\left(f'(z) - \frac{g''(z)}{g'(z)}\right) + 1\right] \\
&\leq |z| \cdot |\alpha| \left[\left|f'(z)\right| + \left|\frac{g''(z)}{g'(z)}\right|\right] + 1 \\
&\leq |z| \cdot |\alpha| \left[\left|f'(z)\left(\frac{z}{f(z)}\right)^\mu\right| \left|\frac{f(z)}{z}\right|^\mu + 1\right] + 1. \qquad (2.18)
\end{aligned}$$

Since f is in the class $B(\mu, \lambda)$, $|f(z)| < M$, from General Schwarz Lemma and from (2.18) , we find that

$$\begin{aligned}
\mathrm{Re}\left[\frac{zJ_\alpha''(f,g)(z)}{J_\alpha'(f,g)(z)} + 1\right] &< |\alpha| \left[1 + \left(\left|f'(z)\left(\frac{z}{f(z)}\right)^\mu - 1\right| + 1\right) M^\mu\right] + 1 \\
&< |\alpha| \left[1 + (2 - \lambda) M^\mu\right] + 1 = \rho. \qquad (2.19)
\end{aligned}$$

We yield that the function $J_\alpha(f, g)$ is in the class $N(\rho)$. □

For $\mu = 0$ in Theorem 2.4 we obtain:

Corollary 2.5. *Let the functions $f, g \in \mathcal{A}$, with f in the class $R_\lambda, 0 \leq \lambda < 1$, and α a complex number, with $\mathrm{Re}\alpha \geq 1$. If $|f(z)| < M$, for M a positive real number, $M \geq 1$, $z \in U$ and $\left|\frac{g''(z)}{g'(z)}\right| < 1$, then the integral operator $J_\alpha(f, g)$ defined by (1.3) is in the class $N(\rho)$, where $\rho = |\alpha| (3 - \lambda) + 1$.*

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