DOI: 10.1515/auom-2016-0030
An. Şt. Univ. Ovidius Constanţa **Vol. 24(2),2016, 105–126**

Repdigits as Euler functions of Lucas numbers

Jhon J. Bravo, Bernadette Faye, Florian Luca and Amadou Tall

Abstract

We prove some results about the structure of all Lucas numbers whose Euler function is a repdigit in base 10. For example, we show that if L_n is such a Lucas number, then $n < 10^{111}$ is of the form p or p^2, where $p^3 \mid 10^{p-1} - 1$.

1 Introduction

Let $\phi(m)$ be the Euler function of the positive integer m. Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ be the sequence of Fibonacci and Lucas numbers given by $F_0 = 0,\ F_1 = 1$ and $L_0 = 2,\ L_1 = 1$ and recurrences

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n \quad \text{for all} \quad n \geq 0.$$

Various Diophantine equations involving the Euler function of members of Fibonacci and Lucas numbers were investigated (see [6], [8], [9]). In [10], it was shown that $n = 11$ is the largest solution of the Diophantine equation

$$\phi(F_n) = d\left(\frac{10^m - 1}{9}\right) \qquad d \in \{1, \ldots, 9\}. \tag{1}$$

Numbers as the ones appearing in the right–hand side of equation (1) are called *rep-digits* in base 10, since their base 10 representation is the string $\underbrace{dd\cdots d}_{m \text{ times}}$. Here, we look at Diophantine equation (1) with F_n replaced by L_n:

$$\phi(L_n) = d\left(\frac{10^m - 1}{9}\right) \qquad d \in \{1, \ldots, 9\}. \tag{2}$$

Key Words: Fibonacci numbers, Lucas numbers, Applications of linear forms in logarithms.
2010 Mathematics Subject Classification: Primary 11B39; Secondary 11D61.
Received: 30.01.2014
Revised: 25.07.2014
Accepted: 02.03.2015

Theorem 1. *Assume that $n > 6$ is such that equation (2) holds with some d. Then:*

- $d = 8$;
- m *is even;*
- $n = p$ *or* p^2, *where* $p^3 \mid 10^{p-1} - 1$.
- $10^9 < p < 10^{111}$.

2 Preliminaries

We will use the property that $L_u \mid L_v$ whenever $u \mid v$ and v/u is odd. One important property that we will use over and over again is the existence of the primitive divisors for the sequence $\{L_n\}_{n\geq 0}$. To formulate it, a primitive divisor of L_n is a prime factor p of L_n which does not divide L_m for any $1 \leq m < n$.

Lemma 2.1 (Carmichael [5]). *L_n has a primitive divisor for all $n \neq 6$, while $L_6 = 2 \times 3^2$, and $2 \mid L_3$, $3 \mid L_2$.*

A primitive prime factor p of L_n has the property that $p \equiv \left(\frac{p}{5}\right) \pmod{n}$. Here and in what follows, for an integer a and an odd prime p we use $\left(\frac{a}{p}\right)$ for the Legendre symbol of a with respect to p. In particular, if p is primitive for L_n, then $p \equiv 1 \pmod{n}$ if $p \equiv 1, 4 \pmod{5}$, and $p \equiv -1 \pmod{n}$ if $p \equiv 2, 3 \pmod{5}$.

Finally, we will use the fact that there are no perfect powers other than 1, 4 in the Lucas sequence $\{L_n\}_{n\geq 0}$. More precisely, we have the following result.

Lemma 2.2 (Bugeaud, Luca, Mignotte and Siksek, [3] and [4]). *The equation $L_n = y^k$ with some $k \geq 1$ implies that $n \in \{1, 3\}$. Furthermore, the only solutions of the equation $L_n = q^a y^k$ for some prime $q < 1087$ and integers $a > 0$, $k \geq 2$ have $n \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 17\}$.*

We will also need the following result about square-classes of members of Lucas sequences due to McDaniel and Ribenboim.

Lemma 2.3 (MacDaniel and Ribenboim [12]). *If $L_m L_n = \square$ with $n > m \geq 0$, then $(m, n) = (1, 3)$, $(0, 6)$ or $(m, 3m)$ with $3 \nmid m$ odd.*

3 Linear forms in logarithms

Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The logarithmic height of η is given by

$$h(\eta) = \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\} \right).$$

Later in the paper we use the following theorem of Matveev [11].

Theorem 2 (Matveev [11]). *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$ $\eta_1, \ldots, \eta_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda = \eta_1^{b_1} \cdots \eta_t^{b_t} - 1 \qquad \text{and} \qquad B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

Let $A_i \geq \max\{Dh(\eta_i), |\log \eta_i|, 0.16\}$ be real numbers, for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t).$$

4 The Baker–Davenport lemma

In 1998, Dujella and Pethő in [7, Lemma 5(a)] gave a version of the reduction method based on a lemma of Baker–Davenport lemma [1]. We next present the following lemma from [2], which is an immediate variation of the result due to Dujella and Pethő from [7], and will be the key tool used to reduce the upper bound on the variable n when we assume that $n \notin \{p, p^2\}$.

Lemma 4.1. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$, where $|| \cdot ||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < u\gamma - v + \mu < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

5 The proof of Theorem 1

5.1 The exponent of 2 in both sides of (2)

Write

$$L_n = 2^{\delta} p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \tag{3}$$

where $\delta \geq 0$, $r \geq 0$, $p_1, \ldots, p_r$ are distinct odd primes and $\alpha_1, \ldots, \alpha_r$ are positive integers. Then

$$\phi(L_n) = 2^{\max\{0,\delta-1\}} p_1^{\alpha_1-1}(p_1-1) p_2^{\alpha_2-1}(p_2-1) \cdots p_r^{\alpha_r-1}(p_r-1). \tag{4}$$

For a nonzero integer m we write $\mathrm{ord}_2(m)$ for the exponent of 2 in the factorization of m. Applying the ord_2 function in both sides of (2) and using (4), we get

$$\begin{aligned} \max\{0,\delta-1\} + \sum_{i=1}^{r} \mathrm{ord}_2(p_i - 1) &= \mathrm{ord}_2(\phi(L_n)) \\ &= \mathrm{ord}_2\left(d\left(\frac{10^m-1}{9}\right)\right) = \mathrm{ord}_2(d). \end{aligned} \tag{5}$$

Note that $\mathrm{ord}_2(d) \in \{0,1,2,3\}$. Note also that $r \leq 3$ and since L_n is never a multiple of 5, we have that

$$\frac{\phi(L_n)}{L_n} \geq \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{7}\right) > \frac{1}{4}, \tag{6}$$

so $\phi(L_n) > L_n/4$. This shows that if $n \geq 8$ satisfies equation (2), then $\phi(L_n) > L_8/4 > 10$, so $m \geq 2$.

We will also use in the later stages of the paper the Binet formula

$$L_n = \alpha^n + \beta^n \qquad (n \geq 0), \tag{7}$$

where $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$. In particular,

$$L_n - 1 = \alpha^n - (1-\beta^n) \leq \alpha^n \qquad \text{for all} \qquad n \geq 0. \tag{8}$$

Furthermore,

$$\alpha^{n-1} \leq L_n < \alpha^{n+1} \qquad \text{for all} \qquad n \geq 1. \tag{9}$$

5.2 The case of the digit $d \notin \{4,8\}$

If $\mathrm{ord}_2(d) = 0$, we get that $d \in \{1,3,5,7,9\}$, $\phi(L_n)$ is odd, so $L_n \in \{1,2\}$, therefore $n = 0,\ 1$. If $\mathrm{ord}_2(d) = 1$, we get that $d \in \{2,6\}$, and from (5) either

$\delta = 2$ and $r = 0$, so $L_n = 4$, therefore $n = 3$, or $\delta \in \{0, 1\}$, $r = 1$ and $p_1 \equiv 3 \pmod 4$. Thus, $L_n = p_1^{\alpha_1}$ or $L_n = 2p_1^{\alpha_1}$. Lemma 2.2 shows that $\alpha_1 = 1$ except for the case when $n = 6$ when $L_6 = 2 \times 3^2$. So, for $n \neq 6$, we get that $L_n = p_1$ or $2p_1$. Let us see that the second case is not possible. Assuming it is, we get $6 \mid n$. Write $n = 2^t \times 3 \times m$, where $t \geq 1$ and m is odd. Clearly, $n \neq 6$.

If $m > 1$, then $L_{2^t 3m}$ has a primitive divisor which does not divide the number $L_{2^t 3}$. Hence, $L_n = 2p_1$ is not possible in this case. However, if $m = 1$ then $t > 1$, and both L_{2^t} and $L_{2^t 3}$ have primitive divisors, so the equation $L_n = 2p_1$ is not possible in this case either. So, the only possible case is $L_n = p_1$. Thus, we get

$$\phi(L_n) = L_n - 1 = d\left(\frac{10^m - 1}{9}\right) \quad \text{and} \quad d \in \{2, 6\},$$

so

$$L_n = d\left(\frac{10^m - 1}{9}\right) + 1 \quad \text{and} \quad d \in \{2, 6\}.$$

When $d = 2$, we get that $L_n \equiv 3 \pmod 5$. The period of the Lucas sequence $\{L_n\}_{n \geq 0}$ modulo 5 is 4. Furthermore, from $L_n \equiv 3 \pmod 5$, we get that $n \equiv 2 \pmod 4$. Thus, $n = 2(2k + 1)$ for some $k \geq 0$. However, this is not possible for $k \geq 1$, since for $k = 1$, we get that $n = 6$ and $L_6 = 2 \times 3^2$, while for $k > 1$, we have that L_n is divisible by both the primes 3 and at least another prime, namely a primitive prime factor of L_n, so $L_n = p_1$ is not possible. Thus, $k = 0$, so $n = 2$.

When $d = 6$, we get that $L_n \equiv 2 \pmod 5$. This shows that $4 \mid n$. Write $n = 2^t(2k + 1)$ for some $t \geq 2$ and $k \geq 0$. As before, if $k \geq 1$, then L_n cannot be a prime since either $k = 1$, so $3 \mid n$, and then $L_n > 2$ is even, or $k \geq 2$, and then L_n is divisible by at least two primes, namely the primitive prime factors of L_{2^t} and of L_n. Thus, $n = 2^t$. Assuming $m \geq 2$, and reducing both sides of the above formula

$$L_{2^{t-1}}^2 - 2 = L_{2^t} = 6\left(\frac{10^m - 1}{9}\right) + 1$$

modulo 8, we get $7 \equiv -5 \pmod 8$, which is not possible. This shows that $m = 1$, so $t = 2$, therefore $n = 4$.

To summarize, we have proved the following result.

Lemma 5.1. *Equation* (2) *has no solutions with* $n > 6$ *if* $d \notin \{4, 8\}$.

5.3 The case of L_n even

Next we treat the case $\delta > 0$. It is well-known and easy to see by looking at the period of $\{L_n\}_{n \geq 0}$ modulo 8 that $8 \nmid L_n$ for any n. Hence, we only need to deal with the cases $\delta = 1$ or 2.

If $\delta = 2$, then $3 \mid n$ and n is odd. Furthermore, relation (5) shows that $r \leq 2$. Assume first that $n = 3^t$. We check that $t = 2, 3$ are not convenient. For $t \geq 4$, we have that L_9, L_{27} and L_{81} are divisors of L_n and all have odd primitive divisors which are prime factors of L_n, contradicting the fact that $r \leq 2$. Assume now that n is a multiple of some prime $p \geq 5$. Then L_p and L_{3p} already have primitive prime factors, so $n = 3p$, for if not, then $n > 3p$, and L_n would have (at least) one additional prime factor, namely a primitive prime factor of L_n. Thus, $n = 3p$. Write

$$L_n = L_{3p} = L_p(L_p^2 + 3).$$

The two factors above are coprime, so, up to relabeling the prime factors of L_n, we may assume that $L_p = p_1^{\alpha_1}$ and $L_p^2 + 3 = 4p_2^{\alpha_2}$. Lemma 2.2 shows that $\alpha_1 = 1$. Further, since p is odd, we get that $L_p \equiv 1, 4 \pmod 5$, therefore the second relation above implies that $p_2^{\alpha_2} \equiv 1 \pmod 5$. If α_2 is odd, we then get that $p_2 \equiv 1 \pmod 5$. This leads to $5 \mid (p_2 - 1) \mid \phi(L_n) = d(10^m - 1)/9$ with $d \in \{4, 8\}$, which is a contradiction. Thus, α_2 is even, showing that

$$L_p^2 + 3 = \square,$$

which is impossible.

If $\delta = 1$, then $6 \mid n$. Assume first that $p \mid n$ for some prime $p > 3$. Write $n = 2^t \times 3 \times m$. If $t \geq 2$, then $r \geq 4$, since L_n is then a multiple of a primitive prime factor of L_{2^t}, a primitive prime factor of $L_{2^t 3}$, a primitive prime factor of $L_{2^t p}$ and a primitive prime factor of $L_{2^t 3p}$. So, $t = 1$. Then L_n is a multiple of 3 and of the primitive prime factors of L_{2p} and L_{6p}, showing that $n = 6p$, for if not, then $n > 6p$ and L_n would have (at least) an additional prime factor, namely a primitive prime factor of L_n. Thus, with $n = 6p$, we may write

$$L_n = L_{6p} = L_{2p}(L_{2p}^2 - 3).$$

Further, it is easy to see that up to relabeling the prime factors of L_n, we may assume that $p_1 = 3$, $\alpha_1 = 2$, $L_{2p} = 3p_2^{\alpha_2}$ and $L_{2p}^2 - 3 = 6p_3^{\alpha_3}$. Furthermore, since $r = 3$, relation (5) tells us that $p_i \equiv 3 \pmod 4$ for $i = 2, 3$. Reducing equation

$$L_p^2 + 2 = L_{2p} = 3p_2^{\alpha_2}$$

modulo 4 we get $3 \equiv 3^{\alpha_2 + 1} \pmod 4$, so α_2 is even. We thus get $L_{2p} = 3\square$, an equation which has no solutions by Lemma 2.2.

So, it remains to assume that $n = 2^t \times 3^s$.

Assume $s \geq 2$. If also $t \geq 2$, then L_n is divisible by the primitive prime factors of L_{2^t}, $L_{2^t 3}$ and $L_{2^t 9}$. This shows that $n = 2^t \times 9$ and we have

$$L_n = L_{2^t 9} = L_{2^t}(L_{2^t}^2 - 3)(L_{2^t 3}^2 - 3).$$

Up to relabeling the prime factors of L_n, we get $L_{2^t} = p_1^{\alpha_1}$, $L_{2^t}^2 - 3 = 2p_2^{\alpha_2}$, $L_{2^t 3}^2 - 3 = p_3^{\alpha_3}$ and $p_i \equiv 3 \pmod 4$ for $i = 1, 2, 3$. Reducing the last relation modulo 4, we get $1 \equiv 3^{\alpha_3} \pmod 4$, so α_3 is even. We thus get $L_{2^t 3}^2 - 3 = \square$, and this is false. Thus, $t = 1$. By the existence of primitive divisors Lemma 2.1, $s \in \{2, 3\}$, so $n \in \{18, 54\}$ and none leads to a solution.

Assume next that $s = 1$. Then $n = 2^t \times 3$ and $t \geq 2$. We write

$$L_n = L_{2^t 3} = L_{2^t}(L_{2^t}^2 - 3).$$

Assume first that there exist i such that $p_i \equiv 1 \pmod 4$. Then $r \leq 2$ by (5). It then follows that in fact $r = 2$ and up to relabeling the primes we have $L_{2^t} = p_1^{\alpha_1}$ and $L_{2^t}^2 - 3 = 2p_2^{\alpha_2}$. Since $L_{2^t} = L_{2^{t-1}}^2 - 2$, we get that $L_{2^{t-1}}^2 - 2 = p_1^{\alpha_1}$, which reduced modulo 4 gives $3 \equiv p_1^{\alpha_1} \pmod 4$, therefore $p_1 \equiv 3 \pmod 4$. As for the second relation, we get $(L_{2^t}^2 - 3)/2 = p_2^{\alpha_2}$, which reduced modulo 4 also gives $3 \equiv p_2^{\alpha_2} \pmod 4$, so also $p_2 \equiv 3 \pmod 4$. But this contradicts the fact that $p_i \equiv 1 \pmod 4$ for some $i \in \{1, \ldots, r\}$. Thus, $p_i \equiv 3 \pmod 4$ for all $i \in \{1, \ldots, r\}$. Reducing relation

$$L_{2^t 3}^2 - 5F_{2^t 3}^2 = 4$$

modulo p_i, we get that $\left(\dfrac{-5}{p_i}\right) = -1$, and since $p_i \equiv 3 \pmod 4$, we get that $\left(\dfrac{5}{p_i}\right) = -1$ for $i \in \{1, \ldots, r\}$. Since p_i are also primitive prime factors for L_{2^t} and/or $L_{2^t 3}$, respectively, we get that $p_i \equiv -1 \pmod{2^t}$.

Suppose next that $r = 2$. We then get that $d = 4$,

$$L_{2^{t-1}}^2 - 2 = L_{2^t} = p_1^{\alpha_1} \qquad \text{and} \qquad L_{2^t}^2 - 3 = 2p_2^{\alpha_2}.$$

Reducing the above relations modulo 8, we get that α_1, α_2 are odd. Thus,

$$\begin{aligned} 4\left(\frac{10^m - 1}{9}\right) &= \phi(L_n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1) \\ \equiv \quad & (-1)^{\alpha_1 - 1}(-2)(-1)^{\alpha_2 - 1}(-2) \pmod{2^t} \equiv 4 \pmod{2^t}, \end{aligned}$$

giving

$$\frac{10^m - 1}{9} \equiv 1 \pmod{2^{t-2}} \qquad \text{therefore} \qquad 10^m \equiv 10 \pmod{2^{t-2}},$$

so $t \leq 3$ for $m \geq 2$. Thus, $n \in \{12, 24\}$, and none of these values leads to a solution of equation (2).

Assume next that $r = 3$. We then get that $d = 8$ and either

$$L_{2^{t-1}}^2 - 2 = L_{2^t} = p_1^{\alpha_1} p_2^{\alpha_2} \qquad \text{and} \qquad L_{2^t}^2 - 3 = 2p_3^{\alpha_3},$$

or

$$L_{2^{t-1}}^2 - 2 = L_{2^t} = p_1^{\alpha_1} \qquad \text{and} \qquad L_{2^t}^2 - 3 = 2p_2^{\alpha_2}p_3^{\alpha_3}.$$

Reducing the above relations modulo 8 as we did before, we get that exactly one of $\alpha_1, \alpha_2, \alpha_3$ is even and the other two are odd. Then

$$\begin{aligned} & 8\left(\frac{10^m - 1}{9}\right) = \phi(L_n) = p_1^{\alpha_1 - 1}(p_1 - 1)p_2^{\alpha_2 - 1}(p_2 - 1)p_3^{\alpha_3 - 1}(p_3 - 1) \\ \equiv \quad & (-1)^{\alpha_1+\alpha_2+\alpha_3-3}(-2)^3 \pmod{2^t} \equiv 8 \pmod{2^t} \end{aligned}$$

giving

$$\frac{10^m - 1}{9} \equiv 1 \pmod{2^{\max\{0,t-3\}}} \quad \text{therefore} \quad 10^m \equiv 10 \pmod{2^{\max\{0,t-3\}}},$$

which implies that $t \leq 4$ for $m \geq 2$. The only new possibility is $n = 48$, which does not fulfill (2).

So, we proved the following result.

Lemma 5.2. *There is no $n > 6$ with L_n even such that relation* (2) *holds.*

5.4 The case of n even

Next we look at solutions of (2) with n even. Write $n = 2^t m$, where $t \geq 1$, m is odd and coprime to 3.

Assume first that there exists i such that $p_i \equiv 1 \pmod 4$. Without loss of generality we assume that $p_1 \equiv 1 \pmod 4$. It then follows from (5) that $r \leq 2$, and that $r = 1$ if $d = 4$. So, if $d = 4$, then $r = 1$, $L_n = p_1^{\alpha_1}$, and by Lemma 2.2, we get that $\alpha_1 = 1$. In this case, by the existence of primitive divisors Lemma 2.1, we get that $m = 1$, otherwise L_n would be divisible both by a primitive prime factor of L_{2^t} as well as by a primitive prime factor of L_n. Hence, $L_{2^t} = p_1$, so

$$L_{2^t} - 1 = \phi(L_{2^t}) = 4\left(\frac{10^m - 1}{9}\right), \quad \text{therefore} \quad L_{2^t} \equiv 5 \pmod{10}.$$

Thus, $5 \mid L_n$ and this is not possible for any n. Suppose now that $d = 8$. If $t \geq 2$, then

$$L_{n/2}^2 - 2 = L_n$$

and reducing the above relation modulo p_1, we get that $\left(\frac{2}{p_1}\right) = 1$. Since $p_1 \equiv 1 \pmod 4$, we read that $p_1 \equiv 1 \pmod 8$. Relation (5) shows that $r = 1$

so $L_n = p_1^{\alpha_1}$. By Lemma 2.2, we get again that $\alpha_1 = 1$ and by the existence of primitive divisors Lemma 2.1, we get that $m = 1$. Thus,

$$L_{2^t} - 1 = \phi(L_{2^t}) = 8\left(\frac{10^m - 1}{9}\right), \quad \text{therefore} \quad L_{2^t} \equiv 4 \pmod 5,$$

which is impossible for $t \geq 2$, since $L_n \equiv 2 \pmod 5$ whenever n is a multiple of 4. This shows that $t = 1$, so $m > 1$. Let $p \geq 5$ be a prime factor of n. Then L_n is divisible by 3 and by the primitive prime factor of L_{2p}, and since $r \leq 2$, we get that $r = 2$, and $n = 2p$. Thus, $L_n = L_{2p} = 3p_2^{\alpha_2}$, and, by Lemma 2.2, we get that $\alpha_2 = 1$. Reducing the above relation modulo 5, we get that $3 \equiv 3p_2 \pmod 5$, so $p_2 \equiv 1 \pmod 5$, showing that $5 \mid (p_2 - 1) \mid \phi(L_n) = 8(10^m - 1)/9$, which is impossible.

This shows that in fact we have $p_i \equiv 3 \pmod 4$ for $i = 1, \ldots, r$. Reducing relation $L_n^2 - 5F_n^2 = 4$ modulo p_i, we get that $\left(\frac{-5}{p_i}\right) = 1$ for $i = 1, \ldots, r$. Since we already know that $\left(\frac{-1}{p_i}\right) = -1$, we get that $\left(\frac{5}{p_i}\right) = -1$ for all $i = 1, \ldots, r$. Since in fact p_i is always a primitive divisor for $L_{2^t d_i}$ for some divisor d_i of m, we get that $p_i \equiv -1 \pmod{2^t}$. Reducing relation

$$L_n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

modulo 4, we get $3 \equiv 3^{\alpha_1 + \cdots + \alpha_r} \pmod 4$, therefore $\alpha_1 + \cdots + \alpha_r$ is odd. Next, reducing the relation

$$\phi(L_n) = p_1^{\alpha_1 - 1}(p_1 - 1) \cdots p_r^{\alpha_r - 1}(p_r - 1)$$

modulo 2^t, we get

$$d\left(\frac{10^m - 1}{9}\right) = \phi(L_n) \equiv (-1)^{\alpha_1 + \cdots + \alpha_r - r}(-2)^r \pmod{2^t} \equiv -2^r \pmod{2^t}.$$

Since $r \in \{2, 3\}$ and $d = 2^r$, we get that

$$\frac{10^m - 1}{9} \equiv -1 \pmod{2^{\max\{0, t-r\}}}, \quad \text{so} \quad 10^m \equiv 8 \pmod{2^{\max\{0, t-r\}}}.$$

Thus, if $m \geq 4$, then $t \leq 6$. Suppose that $m \geq 4$. Computing L_{2^t} for $t \in \{5, 6\}$, we get that each of them has a prime factor p such that $p \equiv 1 \pmod 5$. Thus, $5 \mid (p - 1) \mid \phi(L_n) = d(10^m - 1)/9$, which is impossible. Hence, $t \in \{1, 2, 3, 4\}$. We get the relations

$$L_{2^t m} = L_{2^t} p_1^{\alpha_1}, \quad \text{or} \quad L_{2^t m} = L_{2^t} p_2^{\alpha_2} p_3^{\alpha_3} \quad \text{and} \quad t \in \{1, 2, 3, 4\}. \tag{10}$$

Assume that the left relation (10) holds for some $t \in \{1,2,3,4\}$. Reducing the left equation (10) modulo 5, we get that $L_{2^t} \equiv L_{2^t} p_1^{\alpha_1} \pmod 5$, therefore $p_1^{\alpha_1} \equiv 1 \pmod 5$. If α_1 is odd, we then get that $p_1 \equiv 1 \pmod 5$; hence, $5 \mid (p_1 - 1) \mid \phi(L_n) = d(10^m - 1)/9$ with $d \in \{4,8\}$, which is impossible. If α_1 is even, we then get that $L_n/L_{2^t} = p_1^{\alpha_1} = \square$, and this is impossible since $n \neq 2^t \times 3$ by Lemma 2.3. Assume now that the right relation (10) holds for some $t \in \{2,3,4\}$. Reducing it modulo 5, we get $L_{2^t} \equiv L_{2^t} p_2^{\alpha_2} p_3^{\alpha_3} \pmod 5$. Hence, $p_2^{\alpha_2} p_3^{\alpha_3} \equiv 1 \pmod 5$. Now

$$\begin{aligned} 8\left(\frac{10^m-1}{9}\right) &= \phi(L_n) = (L_{2^t}-1)p_2^{\alpha_2-1}p_3^{\alpha_3-1}(p_2-1)(p_3-1) \\ &\equiv \left(\frac{p_2-1}{p_2}\right)\left(\frac{p_3-1}{p_3}\right) \pmod 5, \end{aligned}$$

so

$$\left(\frac{p_2-1}{p_2}\right)\left(\frac{p_3-1}{p_3}\right) \equiv 3 \pmod 5.$$

The above relation shows that p_2 and p_3 are distinct modulo 5, because otherwise the left–hand side above is a quadratic residue modulo 5 while 3 is not a quadratic residue modulo 5. Thus, $\{p_2, p_3\} \equiv \{2,3\} \pmod 5$, and we get

$$\left(\frac{2-1}{2}\right)\left(\frac{3-1}{3}\right) \equiv 3 \pmod 5 \quad \text{or} \quad 1 \equiv 3^2 \pmod 5,$$

a contradiction. Finally, assume that $t = 1$ and that the right relation (10) holds. Reducing it modulo 4, we get $3 \equiv 3^{\alpha_2+\alpha_3} \pmod 4$, therefore $\alpha_2 + \alpha_3$ is even. If α_2 is even, then so is α_3, so we get that $L_{2m} = 3\square$, which is false by Lemma 2.3. Hence, α_2 and α_3 are both odd. Furthermore, since m is odd and not a multiple of 3, we get that $2m \equiv 2 \pmod 4$ and $2m \equiv 2,4 \pmod 6$, giving $2m \equiv 2, 10 \pmod{12}$. The period of $\{L_n\}_{n\geq 1}$ modulo 8 is 12, and $L_2 \equiv L_{10} \equiv 3 \pmod 8$, showing that $L_{2m} \equiv 3 \pmod 8$. This shows that $p_2^{\alpha_2} p_3^{\alpha_3} \equiv 1 \pmod 8$, and since α_2 and α_3 are odd, we get the congruence $p_2 p_3 \equiv 1 \pmod 8$. This together with the fact that $p_i \equiv 3 \pmod 4$ for $i = 1, 2$, implies that $p_2 \equiv p_3 \pmod 8$. Thus, $(p_2 - 1)/2$ and $(p_3 - 1)/2$ are congruent modulo 4 so their product is 1 modulo 4. Now we write

$$\begin{aligned} \phi(L_n) &= (3-1)(p_2-1)p_2^{\alpha_2-1}(p_3-1)p_3^{\alpha_3-1} \\ &= 8\left(\frac{(p_2-1)}{2}\frac{(p_3-1)}{2}\right)p_2^{\alpha_2-1}p_3^{\alpha_3-1} = 8M, \end{aligned}$$

where $M \equiv 1 \pmod 4$. However, since in fact $M = (10^m - 1)/9$, we get that $M \equiv 3 \pmod 4$ for $m \geq 2$, a contradiction. So, we must have $m \leq 3$, therefore $L_n < 4000$, so $n \leq 17$, and such values can be dealt with by hand.

Thus, we have proved the following result.

Lemma 5.3. *There is no $n > 6$ even such that relation* (2) *holds.*

5.5 $r = 3,\ d = 8$ and m is even

From now on, $n > 6$ is odd and L_n is also odd. If p is any prime factor of L_n, then reducing the equation $L_n^2 - 5F_n^2 = -4$ modulo p we get that $\left(\frac{5}{p}\right) = 1$. Thus, $p \equiv 1, 4 \pmod 5$. If $p \equiv 1 \pmod 5$, then $5 \mid (p-1) \mid \phi(L_n) = d(10^m - 1)/9$ with $d \in \{4, 8\}$, a contradiction. Thus, $p_i \equiv 4 \pmod 5$ for all $i = 1, \ldots, r$.

We next show that $p_i \equiv 3 \pmod 4$ for all $i = 1, \ldots, r$. Assume that this is not so and suppose that $p_1 \equiv 1 \pmod 4$. If $r = 1$, then $L_n = p_1^{\alpha_1}$ and by Lemma 2.2, we have $\alpha_1 = 1$. So,

$$L_n - 1 = \phi(L_n) = d\left(\frac{10^m - 1}{9}\right) \quad \text{so} \quad L_n = d\left(\frac{10^m - 1}{9}\right) + 1.$$

If $d = 4$, then $L_n \equiv 5 \pmod{10}$, so $5 \mid L_n$, which is false. When $d = 8$, we get that $L_n \equiv 4 \pmod 5$, showing that $n \equiv 3 \pmod 4$. However, we also have that $L_n \equiv 1 \pmod 8$, showing that $n \equiv 1 \pmod{12}$; in particular, $n \equiv 1 \pmod 4$, a contradiction.

Assume now that $r = 2$. Then $L_n = p_1^{\alpha_1} p_2^{\alpha_2}$ and $d = 8$. Then

$$\phi(L_n) = (p_1 - 1)p_1^{\alpha_1 - 1}(p_2 - 1)p_2^{\alpha_2 - 1} = 8\left(\frac{10^m - 1}{9}\right). \tag{11}$$

Reducing the above relation (11) modulo 5 we get $4^{\alpha_1 + \alpha_2 - 2} \times 3^2 \equiv 3 \pmod 5$, which is impossible since the left–hand side of it is a quadratic residue modulo 5 while the right–hand side of it is not.

Thus, $p_i \equiv 3 \pmod 4$ for $i = 1, \ldots, r$. Assume next that $r = 2$. Then $L_n = p_1^{\alpha_1} p_2^{\alpha_2}$ and $d = 4$. Then

$$\phi(L_n) = (p_1 - 1)p_1^{\alpha_1 - 1}(p_2 - 1)p_2^{\alpha_2 - 1} = 4\left(\frac{10^m - 1}{9}\right). \tag{12}$$

Reducing the above relation (12) modulo 5, we get $4^{\alpha_1 + \alpha_2 - 2} \times 3^2 \equiv 4 \pmod 5$, therefore $4^{\alpha_1 + \alpha_2 - 2} \equiv 1 \pmod 5$. Thus, $\alpha_1 + \alpha_2$ is even. If α_1 is even, so is α_2, so $L_n = \square$, and this is false by Lemma 2.2. Hence, α_2 and α_3 are both odd. It now follows that $L_n \equiv 3^{\alpha_1 + \alpha_2} \pmod 4$, so $L_n \equiv 1 \pmod 4$, therefore $n \equiv 1 \pmod 6$, and also $L_n \equiv 4^{\alpha_1 + \alpha_2} \pmod 5$, so $L_n \equiv 1 \pmod 5$, showing that $n \equiv 1 \pmod 4$. Hence, $n \equiv 1 \pmod{12}$, showing that $L_n \equiv 1 \pmod 8$. Thus, $p_1^{\alpha_1} p_2^{\alpha_2} \equiv 1 \pmod 8$, and since α_1 and α_2 are odd and $p_1^{\alpha_1 - 1}$ and $p_2^{\alpha_2 - 1}$ are congruent to 1 modulo 8 (as perfect squares), we therefore get that $p_1 p_2 \equiv 1$

(mod 8). Since also $p_1 \equiv p_2 \equiv 3 \pmod 4$, we get that in fact $p_1 \equiv p_2 \pmod 8$. Thus, $(p_1-1)/2$ and $(p_2-1)/2$ are congruent modulo 4 so their product is 1 modulo 4. Thus,

$$\phi(L_n) = 4\left(\frac{(p_1-1)}{2}\frac{(p_2-1)}{2}\right)p_1^{\alpha_1-1}p_2^{\alpha_2-1} = 4M,$$

where $M \equiv 1 \pmod 4$. Since in fact we have $M = (10^m-1)/9$, we get that $M \equiv 3 \pmod 4$ for $m \geq 2$, a contradiction.

Thus, $r = 3$ and $d = 8$. To get that m is even, we write $L_n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$. So,

$$\phi(L_n) = (p_1-1)p_1^{\alpha_1-1}(p_2-1)p_2^{\alpha_2-1}(p_3-1)p_3^{\alpha_3-1} = 8\left(\frac{10^m-1}{9}\right), \quad (13)$$

Reducing equation (13) modulo 5 we get $4^{\alpha_1+\alpha_2+\alpha_3-3}\times 3^3 \equiv 3 \pmod 5$, giving $4^{\alpha_1+\alpha_2+\alpha_3} \equiv 1 \pmod 5$. Hence, $\alpha_1+\alpha_2+\alpha_3$ is even. It is not possible that all α_i are even for $i = 1, 2, 3$, since then we would get $L_n = \square$, which is not possible by Lemma 2.2. Hence, exactly one of them is even, say α_3 and the other two are odd. Then $L_n \equiv 3^{\alpha_1+\alpha_2+\alpha_3} \equiv 1 \pmod 4$ and $L_n \equiv 4^{\alpha_1+\alpha_2+\alpha_3} \equiv 1 \pmod 5$. Thus, $n \equiv 1 \pmod 6$ and $n \equiv 1 \pmod 4$, so $n \equiv 1 \pmod{12}$. This shows that $L_n \equiv 1 \pmod 8$. Since $p_1^{\alpha_1-1}p_2^{\alpha_2-1}p_3^{\alpha_3}$ is congruent to 1 modulo 8 (as a perfect square), we get that $p_1p_2 \equiv 1 \pmod 8$. Thus, $p_1 \equiv p_2 \pmod 8$, so $(p_1-1)/2$ and $(p_2-1)/2$ are congruent modulo 4 so their product is 1. Then

$$\phi(L_n) = 8\left(\frac{(p_1-1)}{2}\frac{(p_2-1)}{2}\right)\left(\frac{p_3(p_3-1)}{2}\right)p_1^{\alpha_1-1}p_2^{\alpha_2-1}p_3^{\alpha_3-2} = 8M, \quad (14)$$

where $M = (10^m-1)/9 \equiv 3 \pmod 4$. In the above product, all odd factors are congruent to 1 modulo 4 except possibly for $p_3(p_3-1)/2$. This shows that $p_3(p_3-1)/2 \equiv 3 \pmod 4$, which shows that $p_3 \equiv 3 \pmod 8$. Now since $p_3^2 \mid L_n$, we get that $p_3 \mid \phi(L_n) = 8(10^m-1)/9$. So, $10^m \equiv 1 \pmod{p_3}$. Assuming that m is odd, we would get

$$1 = \left(\frac{10}{p_3}\right) = \left(\frac{2}{p_3}\right)\left(\frac{5}{p_3}\right) = -1,$$

a contradiction. In the above, we used that $p_3 \equiv 3 \pmod 8$ and $p_3 \equiv 4 \pmod 5$ and quadratic reciprocity to conclude that $\left(\frac{2}{p_3}\right) = -1$ as well as $\left(\frac{5}{p_3}\right) = \left(\frac{p_3}{5}\right) = 1$.

So, we have showed the following result.

Lemma 5.4. *If $n > 6$ is a solution of* (2)*, then n is odd, L_n is odd, $r = 3$, $d = 8$ and m is even. Further, $L_n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, where $p_i \equiv 3 \pmod 4$ and $p_i \equiv 4 \pmod 5$ for $i = 1, 2, 3$, $p_1 \equiv p_2 \pmod 8$, $p_3 \equiv 3 \pmod 8$, α_1 and α_2 are odd and α_3 is even.*

5.6 $n \in \{p, p^2\}$ for some prime p with $p^3 \mid 10^{p-1} - 1$

The factorizations of all Lucas numbers L_n for $n \leq 1000$ are known. We used them and Lemma 5.4 and found no solution to equation (2) with $n \in [7, 1000]$.

Let p be a prime factor of n. Suppose first that $n = p^t$ for some positive integer t. If $t \geq 4$, then L_n is divisible by at least four primes, namely primitive prime factors of L_p, L_{p^2}, L_{p^3} and L_{p^4}, respectively, which is false. Suppose that $t = 3$. Write

$$L_n = L_p \left(\frac{L_{p^2}}{L_p} \right) \left(\frac{L_{p^3}}{L_{p^2}} \right).$$

The three factors above are coprime, so they are $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, $p_3^{\alpha_3}$ in some order. Since α_3 is even, we get that one of L_p, L_{p^2}/L_p or L_{p^3}/L_{p^2} is a square, which is false by Lemmas 2.2 and 2.3. Hence, $n \in \{p,\ p^2\}$. All primes p_1, p_2, p_3 are quadratic residues modulo 5. When $n = p$, they are primitive prime factors of L_p. When $n = p^2$, all of them are primitive prime factors of L_p or L_{p^2} with at least one of them being a primitive prime factor of L_{p^2}. Thus, $p_i \equiv 1 \pmod p$ holds for all $i = 1, 2, 3$ both in the cace $n = p$ and $n = p^2$, and when $n = p^2$ at least one of the the above congruences holds modulo p^2. This shows that $p^3 \mid (p_1 - 1)(p_2 - 1)(p_3 - 1) \mid \phi(L_n) = 8(10^m - 1)/9$, so $p^3 \mid 10^m - 1$. When $n = p^2$, we in fact have $p^4 \mid 10^m - 1$. Assume now that $p^3 \nmid 10^{p-1} - 1$. Then the congruence $p^3 \mid 10^m - 1$ implies $p \mid m$, while the congruence $p^4 \mid 10^m - 1$ implies $p^2 \mid m$. Hence, when $n = p$, we have

$$2^p > L_p > \phi(L_n) = 8(10^m - 1)/9 > (10^p - 1)/9 > 10^{p-1}$$

which is false for any $p \geq 3$. Similarly, if $n = p^2$, then

$$2^{p^2} > L_{p^2} > \phi(L_n) = 8(10^m - 1)/9 > (10^{p^2} - 1)/9 > 10^{p^2-1}$$

which is false for any $p \geq 3$. So, indeed when n is a power of a prime p, then the congruence $p^3 \mid 10^{p-1} - 1$ must hold. We record this as follows.

Lemma 5.5. *If $n > 6$ and $n = p^t$ is solution of* (2) *with some $t \geq 1$ and p prime, then $t \in \{1, 2\}$ and $p^3 \mid 10^{p-1} - 1$.*

Suppose now that n is divisible by two distinct primes p and q. By Lemma 2.1, L_p, L_q and L_{pq} each have primitive prime factors. This shows that $n = pq$, for if $n > pq$, then L_n would have (at least) one additional prime factor, which is a contradiction. Assume $p < q$ and

$$L_n = L_p L_q \left(\frac{L_{pq}}{L_p L_q}\right).$$

Unless $q = L_p$, the three factors above are coprime. Say $q \neq L_p$. Then the three factors above are $p_1^{\alpha_1}$, $p_2^{\alpha_2}$ and $p_3^{\alpha_3}$ in some order. By Lemmas 2.2 and up to relabeling the primes p_1 and p_2, we may assume that $\alpha_1 = \alpha_2 = 1$, so $L_p = p_1$, $L_q = p_2$ and $L_{pq}/(L_p L_q) = p_3^{\alpha_3}$. On the other hand, if $q = L_p$, then $q^2 \| L_{pq}$. This shows then that up to relabeling the primes we may assume that $\alpha_2 = 1$, $\alpha_3 = 2$, $L_p = p_3$, $L_q = p_2$, $L_{pq}/(L_p L_q) = p_3 p_1^{\alpha_1}$. However, in this case $p_3 \equiv 3 \pmod 8$, showing that $p \equiv 5 \pmod 8$. In particular, we also have $p \equiv 1 \pmod 4$, so $p_3 = L_p \equiv 1 \pmod 5$, and this is not possible. So, this case cannot appear.

Write $m = 2m_0$. Then

$$(p_1 - 1)(p_2 - 1)(p_3 - 1)p_3^{\alpha_3 - 1} = \phi(L_n) = \frac{8(10^{m_0} - 1)(10^{m_0} + 1)}{9}.$$

If m_0 is even, then $p_3^{\alpha_3-1} \mid 10^{m_0} - 1$ because $p_3 \equiv 3 \pmod 4$, so p_3 cannot divide $10^{m_0} + 1 = (10^{m_0/2})^2 + 1$. If m_0 is odd, then $p_3^{\alpha_3-1} \mid 10^{m_0} + 1$, because if not we would have that $p_3 \mid 10^{m_0} - 1$, so $10^{m_0} \equiv 1 \pmod{p_3}$, and since m_0 is odd we would get $\left(\frac{10}{p_3}\right) = 1$, which is false since $\left(\frac{2}{p_3}\right) = -1$ and $\left(\frac{5}{p_3}\right) = 1$. Thus, we get, using (8), that

$$\begin{aligned} \alpha^{p+q} p_3 &> (L_p - 1)(L_q - 1)p_3 = p_1 p_2 p_3 > (p_1 - 1)(p_2 - 1)(p_3 - 1) \\ &\geq \frac{8(10^{m_0} - 1)}{9} > \frac{8}{10} \times 10^{m_0}. \end{aligned} \tag{15}$$

On the other hand, by inequality (6), we have

$$10^m > \frac{8(10^m - 1)}{9} = \phi(L_n) > \frac{L_n}{4},$$

so that

$$10^{m_0} > \frac{\sqrt{L_n}}{2} > \frac{\alpha^{pq/2 - 0.5}}{2}, \tag{16}$$

where we used the inequality (9). From (15) and (16), we get

$$p_3 > \frac{8}{20\sqrt{\alpha}} \alpha^{pq/2 - p - q} = \frac{8}{20\alpha^{4.5}} \alpha^{(p-2)(q-2)} > \frac{\alpha^{(p-2)(q-2)}}{25}.$$

Once checks that the inequality

$$\frac{\alpha^{(p-2)(q-2)/2}}{25} > \alpha^{q+1} \tag{17}$$

is valid for all pairs of primes $5 \le p < q$ with $pq > 100$. Indeed, the above inequality (17) is implied by

$$(p-2)(q-2)/2 - (q+1) - 7 > 0, \quad \text{or} \quad (q-2)(p-4) > 20. \tag{18}$$

If $p \ge 7$, then $q > p \ge 11$ and the above inequality (18) is clear, whereas if $p = 5$, then $q \ge 23$ and the inequality (18) is again clear.

We thus get that

$$p_3 > \frac{\alpha^{(p-2)(q-2)}}{25} > \alpha^{q+1} > L_q = p_2 > L_p = p_1.$$

We exploit the two relations

$$\begin{aligned} 0 \quad < \quad 1 - \frac{\phi(L_n)}{L_n} &= 1 - \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) < \frac{3}{p_1} < \frac{5}{\alpha^p}; \\ 1 - \frac{(L_p-1)\phi(L_n)}{L_p L_n} &= 1 - \left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) < \frac{2}{p_2} < \frac{4}{\alpha^q}. \end{aligned} \tag{19}$$

In the above, we used the inequality

$$1 - (1-x_1)\cdots(1-x_r) \le x_1 + \cdots + x_r$$

valid for all real numbers $x_i \in (0,1)$ for $i = 1, \dots, r$, which can be easily proved by induction on r. Since n is odd, we have $L_n = \alpha^n - \alpha^{-n}$. Then

$$1 + \frac{2}{\alpha^{2n}} > \frac{1}{1 - \alpha^{-2n}} > 1,$$

so

$$\frac{1}{\alpha^n} + \frac{2}{\alpha^{3n}} > \frac{1}{L_n} > \frac{1}{\alpha^n},$$

or

$$\frac{8 \times 10^m}{9\alpha^n} + \frac{16 \times 10^m}{9\alpha^{3n}} - \frac{8}{9L_n} > \frac{8(10^m - 1)}{9L_n} = \frac{\phi(L_n)}{L_n} > \frac{8 \times 10^m}{9\alpha^n} - \frac{8}{9L_n}. \tag{20}$$

The first inequality (19) and (20) show that

$$\left|1 - (8/9) \times 10^m \times \alpha^{-n}\right| < \frac{3}{p_1} + \frac{8}{9L_n} + \frac{16 \times 10^m}{9\alpha^{3n}}. \tag{21}$$

Now

$$8 \times 10^{m-1} < \frac{8(10^m - 1)}{9} = \phi(L_n) < L_n < \alpha^{n+1}, \quad \text{so} \quad 10^m < \frac{10\alpha}{8}\alpha^n,$$

showing that

$$\frac{16 \times 10^m}{9\alpha^{3n}} < \frac{20\alpha}{9\alpha^{2n}} < \frac{0.5}{\alpha^n} \quad \text{for} \quad n > 1000.$$

Since also

$$\frac{8}{9L_n} < \frac{8\alpha}{9\alpha^n} < \frac{1.5}{\alpha^n},$$

we get that

$$\frac{16 \times 10^m}{9\alpha^{3n}} + \frac{8}{9L_n} < \frac{0.5}{\alpha^n} + \frac{1.5}{\alpha^n} < \frac{2}{\alpha^n}.$$

Since also $p_1 < L_n^{1/3} < \alpha^{(n+1)/3}$, we get that (21) becomes

$$\left|1 - (8/9) \times 10^m \times \alpha^{-n}\right| < \frac{3}{p_1} + \frac{2}{\alpha^n} < \frac{4}{p_1} = \frac{4}{L_p} < \frac{4\alpha}{\alpha^p} < \frac{7}{\alpha^p}, \tag{22}$$

where the middle inequality is implied by $\alpha^n > 2\alpha^{(n+1)/3} > 13p_1$, which holds for $n > 1000$.

The same argument based on (20) shows that

$$\left|1 - \left(\frac{8(L_p - 1)}{9L_p}\right) \times 10^m \times \alpha^{-n}\right| < \frac{4}{\alpha^q} + \frac{2}{\alpha^n} < \frac{5}{\alpha^q}. \tag{23}$$

We are in a situation to apply Theorem 2 to the left–hand sides of (22) and (23). The expressions there are nonzero, since any one of these expressions being zero means $\alpha^n \in \mathbb{Q}$ for some positive integer n, which is false. We always take $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ for which $D = 2$. We take $t = 3$, $\alpha_1 = \alpha,\ \alpha_2 = 10$, so we can take $A_1 = \log\alpha = 2h(\alpha_1)$ and $A_2 = 2\log 10$. For (22), we take $\alpha_3 = 8/9$, and $A_3 = 2\log 9 = 2h(\alpha_3)$. For (23), we take $\alpha_3 = 8(L_p - 1)/9L_p$, so we can take $A_3 = 2p > h(\alpha_3)$. This last inequality holds because $h(\alpha_3) \le \log(9L_p) < (p+1)\log\alpha + \log 9 < p$ for all $p \ge 7$, while for $p = 5$ we have $h(\alpha_3) = \log 99 < 5$. We take $\alpha_1 = -n,\ \alpha_2 = m,\ \alpha_3 = 1$. Since

$$2^n > L_n > \phi(L_n) > 8 \times 10^{m-1}$$

it follows that $n > m$. So, $B = n$. Now Theorem 2 implies that a lower bound on the left–hand side of (22) is

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log n)(\log\alpha)(2\log 10)(2\log 9)\right),$$

so inequality (22) implies

$$p \log \alpha - \log 7 < 9.5 \times 10^{12}(1 + \log n),$$

which implies

$$p < 2 \times 10^{13}(1 + \log n). \tag{24}$$

Now Theorem 2 implies that the right–hand side of inequality (23) is at least as large as

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log n)(\log \alpha)(2 \log 10)(2p)\right)$$

leading to

$$q \log \alpha - \log 4 < 4.3 \times 10^{12}(1 + \log n)p.$$

Using (24), we get

$$q < 9 \times 10^{12}(1 + \log n)p < 2 \times 10^{26}(1 + \log n)^2.$$

Using again (24), we get

$$n = pq < 4 \times 10^{39}(1 + \log n)^2,$$

leading to

$$n < 5 \times 10^{43}. \tag{25}$$

Now we need to reduce the bound. We return to (22). Put

$$\Lambda = m \log 10 - n \log \alpha + \log(8/9).$$

Then (22) implies that

$$|e^{\Lambda} - 1| < \frac{7}{\alpha^p}. \tag{26}$$

Assuming $p \geq 7$, we get that the right–hand side of (26) is $< 1/2$. Analyzing the cases $\Lambda > 0$ and $\Lambda < 0$ and using the fact that $1 + x < e^x$ holds for all positive real numbers x, we get that

$$|\Lambda| < \frac{14}{\alpha^p}.$$

Assume say that $\Lambda > 0$. Dividing across by $\log \alpha$, we get

$$0 < m\left(\frac{\log 10}{\log \alpha}\right) - n + \left(\frac{\log(8/9)}{\log \alpha}\right) < \frac{30}{\alpha^p}.$$

We are now ready to apply Lemma 4.1 with the obvious parameters

$$\gamma = \frac{\log 10}{\log \alpha}, \quad \mu = \frac{\log(8/9)}{\log \alpha}, \quad A = 30, \quad B = \alpha.$$

Since $m < n$, we can take $M = 10^{45}$ by (25). Applying Lemma 4.1, performing the calculations and treating also the case when $\Lambda < 0$, we get that $p < 250$. Now we go to inequality (23) and for $p \in [5, 250]$, we consider

$$\Lambda_p = m \log 10 - n \log \alpha + \log \left(\frac{8(L_p - 1)}{9L_p} \right).$$

Then inequality (23) becomes

$$\left| e^{\Lambda_p} - 1 \right| < \frac{5}{\alpha^q}. \tag{27}$$

Since $q \geq 7$, the right–hand side is smaller than $1/2$. We thus get that

$$|\Lambda_p| < \frac{10}{\alpha^q}.$$

We proceed in the same way as we proceeded with Λ by applying Lemma 4.1 to Λ_p and distinguishing the cases in which $\Lambda_p > 0$ and $\Lambda_p < 0$, respectively. In all cases, we get that $q < 250$. Thus, $5 \leq p < q < 250$. Note however that we must have either $p^2 \mid 10^{p-1} - 1$ or $q^2 \mid 10^{q-1} - 1$. Indeed, the point is that since all three prime factors of L_n are quadratic residues modulo 5, and they are primitive prime factors of L_p, L_q and L_{pq}, respectively, it follows that $p_1 \equiv 1 \pmod p$, $p_2 \equiv 1 \pmod q$ and $p_3 \equiv 1 \pmod{pq}$. Thus, $(pq)^2 \mid (p_1 - 1)(p_2 - 1)(p_3 - 1) \mid \phi(L_n) = 8(10^m - 1)/9$, which in turn shows that $(pq)^2 \mid 10^m - 1$. Assume that neither $p^2 \mid 10^{p-1} - 1$ nor $q^2 \mid 10^{q-1} - 1$. Then relation $(pq)^2 \mid 10^m - 1$ implies that $pq \mid m$. Thus, $m \geq pq$, leading to

$$2^{pq} > L_n > \phi(L_n) = \frac{8(10^m - 1)}{9} > 10^{m-1} \geq 10^{pq-1},$$

a contradiction. So, indeed either $p^2 \mid 10^{p-1} - 1$ or $q^2 \mid 10^{q-1} - 1$. However, a computation with Mathematica revealed that there is no prime r such that $r^2 \mid 10^{r-1} - 1$ in the interval $[5, 250]$. In fact, the first such $r > 3$ is $r = 487$, but L_{487} is not prime!

This contradiction shows that indeed when $n > 6$, we cannot have $n = pq$. Hence, $n \in \{p, p^2\}$ and $p^3 \mid 10^{p-1} - 1$. We record this as follows.

Lemma 5.6. *Equation* (2) *has no solution* $n > 6$ *which is not of the form* $n = p$ *or* p^2 *for some prime* p *such that* $p^3 \mid 10^{p-1} - 1$.

5.7 Bounding n

Finally, we bound n. We assume again that $n > 1000$. Equation (3) becomes

$$L_n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}.$$

Throughout this last section, we assume that $p_1 < p_2 < p_3$. First, we bound p_1, p_2 and p_3 in terms of n. Using the first relation of (19), we have that

$$0 < 1 - \frac{\phi(L_n)}{L_n} = 1 - \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) < \frac{3}{p_1}. \tag{28}$$

By the argument used at estimates (20)–(22), we get that

$$|1 - (8/9) \times 10^m \times \alpha^{-n}| < \frac{3}{p_1} + \frac{2}{\alpha^n} < \frac{4}{p_1}, \tag{29}$$

where the last inequality holds because $p_1 \le L_n/(p_2p_3) < L_n/(7\times 11) < \alpha^n/2$.

We apply Theorem 2 to the left-hand side of (29) The expression there is nonzero by a previous argument. We take again $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ for which $D = 2$. We take $t = 3$, $\alpha_1 = 8/9$, $\alpha_2 = 10$ and $\alpha_3 = \alpha$. Thus, we can take $A_1 = \log 9 = 2h(\alpha_1)$, $A_2 = 2\log 10$ and $A_3 = 2\log\alpha = 2h(\alpha_3)$. We also take $b_1 = 1$, $b_2 = m$, $b_3 = -n$. We already saw that $B = n$. Now Theorem 2 implies that a lower bound on the left–hand side of (29) is at least

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log n)2^3(\log\alpha)(\log 10)(\log 9)\right),$$

so inequality (22) implies

$$\log p_1 - \log 4 < 1.89 \times 10^{13}(1 + \log n),$$

Then we get

$$\log p_1 < 1.9 \times 10^{13}(1 + \log n). \tag{30}$$

We use the same argument to bound p_2. We have

$$0 < 1 - \left(\frac{p_1 - 1}{p_1}\right)\frac{\phi(L_n)}{L_n} = \left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right) < \frac{2}{p_2}.$$

Thus, we get that:

$$\left|1 - \left(\frac{8(p_1 - 1)}{9p_1}\right) \times 10^m\alpha^{-n}\right| < \frac{2}{p_2} + \frac{2}{\alpha^n} < \frac{3}{p_2}, \tag{31}$$

where the last inequality follows again because $p_2 \le L_n/(p_1p_3) < \alpha^n/2$.

We apply Theorem 2 to the left–hand side of (31). We take $t = 3$, $\alpha_1 = 8(p_1 - 1)/(9p_1)$, $\alpha_2 = 10$ and $\alpha_3 = \alpha$, so we take $A_1 = 2\log(9p_1) \ge 2h(\alpha_1)$, $A_2 = 2\log 10$ and $A_3 = 2\log\alpha$. Again $b_1 = -1$, $b_2 = m$, $b_3 = -n$ and $B = n$. Now Theorem 2 implies that a lower bound on the left–hand side of (31) is

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log n)2^3(\log\alpha)\log 10\log(9p_1)\right).$$

Using estimate (30), inequality (32) implies

$$\log p_2 - \log 2 < 1.8 \times 10^{26}(1 + \log n)^2. \tag{32}$$

Using a similar argument, we get

$$\log p_3 - \log 2 < 1.8 \times 10^{39}(1 + \log n)^3. \tag{33}$$

Now can bound n. Equation (3), gives that :

$$\alpha^n + \beta^n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}.$$

Thus,

$$|p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \alpha^{-n} - 1| = \frac{1}{\alpha^{2n}} \tag{34}$$

We can apply Theorem 2, with $t = 4$, $\alpha_1 = p_1$, $\alpha_2 = p_2$, $\alpha_3 = p_3$, and $\alpha_4 = \alpha$. We take $A_1 = 2\log p_1 = 2h(\alpha_1)$, $A_2 = 2\log p_2$, $A_3 = 2\log p_3 = 2h(\alpha_3)$ and $A_4 = 2\log\alpha$. We take $B = n$. Then Theorem 2 implies that a lower bound on the left–hand side of (34) is

$$\exp\left(-1.4 \times 30^7 \times 4^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log n)2^4(\log\alpha)\prod_{i=1}^{3}(\log p_i)\right).$$

Using (34) and inequalities (29), (32), (33), we get

$$n < 8 \times 10^{93}(1 + \log n)^7, \quad \text{so} \quad n < 10^{111}.$$

This gives the upper bound. As for the lower bound, a quick check with Mathematica revealed that the only primes $p < 2\times 10^9$ such that $p^2 \mid 10^{p-1} - 1$ are $p \in \{3, 487, 56598313\}$ and none of these has in fact the stronger property that $p^3 \mid 10^{p-1} - 1$.

Acknowledgements. We thank the referee for a careful reading of the paper and for comments and suggestions which improved its quality. Part of this work was written during a visit of F. L. at the AIMS-Senegal in January of 2014. He thanks the people of this institution for their hospitality. J. J. B. was partially supported by Universidad del Cauca.

References

[1] A. Baker and H. Davenport, "The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$", *Quart. J. Math. Oxford Ser. (2)* **20** (1969), 129–137.

[2] J. J. Bravo and F. Luca, "On a conjecture about repdigits in k–generalized Fibonacci sequences", *Publ. Math. Debrecen*, **82** (2013), 623–639.

[3] Y. Bugeaud, M. Mignotte and S. Siksek, "Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers", *Ann. of Math. (2)* **163** (2006), 969–1018.

[4] Y. Bugeaud, F. Luca, M. Mignotte and S. Siksek, "Almost powers in the Lucas sequence", *J. N. T. Bordeaux* **20** (2008), 555–600.

[5] R. D. Carmichael, "On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$", *Ann. Math. (2)* **15** (1913), 30–70.

[6] M. T. Damir, B. Faye, F. Luca and A. Tall, "Members of Lucas sequences whose Euler function is a power of 2", *Fibonacci Quarterly*, to appear.

[7] A. Dujella and A. Pethő, "A generalization of a theorem of Baker and Davenport", *Quart. J. Math. Oxford Ser. (2)* **49** (1998), 291–306.

[8] F. Luca, "Equations involving arithmetic functions of Fibonacci numbers," *Fibonacci Quart.* **38** (2000), 49–55.

[9] F. Luca, "Euler indicators of binary recurrent sequences", *Collectanea Math.* **53** (2002), 133–156.

[10] F. Luca and M. Mignotte, "$\phi(F_{11}) = 88$", *Divulgaciones Mat.* **14** (2006), 101–106.

[11] E. M. Matveev, "An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers", *Izv. Math.* **64** (2000), 1217–1269.

[12] W. L. McDaniel and P. Ribenboim, "Square-classes in Lucas sequences having odd parameters", *J. Number Theory* **73** (1998), 14–27.

Jhon J. Bravo,
Departamento de Matemáticas,
Universidad del Cauca,
Calle 5 No 4–70, Popayán, Colombia.
Email: jbravo@unicauca.edu.co

Bernadette Faye,
School of Mathematics,
University of the Witwatersrand,
Private Bag X3, Wits 2050, South Africa and AIMS-Sénégal, Km 2 route de Joal (Centre IRD Mbour), BP: 64566 Dakar-Fann, Sénégal.
Email: bernadette@aims-senegal.org

Florian Luca,
The John Knopfmacher Centre for Applied Analysis and Number Theory,
School of Mathematics,
University of the Witwatersrand,
Private Bag X3, Wits 2050, South Africa and
Centro de Ciencias Matemáticas UNAM, Morelia, Mexico
Email: florian.luca@wits.ac.za

Amadou Tall,
AIMS-Sénégal,
Km 2 route de Joal (Centre IRD Mbour),
BP: 64566 Dakar-Fann, Sénégal.
Email: amadou.tall@aims-senegal.org